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Classification of polynomial integrable systems of mixed scalar and vector evolution equations: I

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Abstract

We perform a classification of integrable systems of mixed scalar and vector evolution equations with respect to higher symmetries. We consider polynomial systems that are homogeneous under a suitable weighting of variables. This paper deals with the KdV weighting, the Burgers (or potential KdV or modified KdV) weighting, the Ibragimov–Shabat weighting and two unfamiliar weightings. The case of other weightings will be studied in a subsequent paper. Making an ansatz for undetermined coefficients and using a computer package for solving bilinear algebraic systems, we give the complete lists of second-order systems with a third-order or a fourth-order symmetry and third-order systems with a fifth-order symmetry. For all but a few systems in the lists, we show that the system (or, at least a subsystem of it) admits either a Lax representation or a linearizing transformation. A thorough comparison with recent work of Foursov and Olver is made.

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1. Introduction

The symmetry approach has been proven to be the most efficient integrability test for (1+1)-dimensional nonlinear evolution equations [1–10] (see also a recent review [11]). It is useful in classifying both scalar evolution equations and evolutionary systems of equations (see e.g. [10]). A milestone in this direction is the work of Mikhailov, Shabat and Yamilov [4, 5, 7, 8] on the classification of second-order systems with two components. Their work dealt with a large class of systems that are non-polynomial in general and have a nondegenerate leading part with respect to x-derivatives. They obtained a complete list of systems possessing higher conservation laws, up to some (almost) invertible transformations [7, 8]. Systems with both higher conservation laws and higher symmetries are believed to be integrable by the *inverse scattering method*, for short 'S-integrable' in the terminology of Calogero [12, 13]. The aim of this paper is to extend the classification of Mikhailov *et al* and to make it easily accessible. To be specific, we pursue the following goals with this paper.

- To provide a 'user-friendly' complete list of systems without any freedom of nontrivial transformations. By that the user does not have to find transformations to locate a given system in our list. Trivial scaling parameters are removed. Naturally, this is possible only for a much more restricted class of systems than that considered by Mikhailov *et al*.
- To include systems without higher conservation laws[3], but with higher symmetries, in the classification. Systems of this sort are believed to be linearizable by an appropriate *change of variables* and, if so, said to be 'C-integrable' in the terminology of Calogero [12, 13].
- To allow systems to have a degenerate leading part. This means that the coefficient matrix of leading terms may have a zero eigenvalue.
- To classify systems of higher order (third order, etc).
- To classify systems with more than two components.

Here we mention earlier studies devoted to these extensions, although we do not know any work dealing with all these extensions simultaneously. A rather user-friendly list of integrable systems of second order with two components was presented in [14] (see also a similar list in [15]). Some classifications of 'C-integrable' systems including coupled Burgers-type equations have been reported in [15–18][4]. Classification of integrable coupled KdV-type equations has been performed in [19–22] using the symmetry approach and in [23, 24] using the Painlevé PDE test. Coupled potential KdV (coupled pKdV) equations and coupled modified KdV (coupled mKdV) equations with higher symmetries were listed in [25] (see also [26]). Classification of coupled KdV equations and coupled mKdV equations was studied in connection with Jordan algebras in [27, 28], where the coefficient matrix of leading terms is restricted to the identity. The Painlevé PDE test was applied to coupled higher order nonlinear Schrödinger equations in [29], where integrable coupled mKdV equations and coupled derivative nonlinear Schrödinger (coupled DNLS) equations were obtained. Classification of non-commutative generalizations of integrable systems on an associative algebra was addressed in [30] (see also [31–33] for DNLS-type systems), while vector generalizations of integrable systems were discussed in [34].

In this paper, we investigate evolutionary systems for one scalar unknown $u(x,t)$ and one vector unknown $U(x,t) \equiv (U_1, U_2, \ldots, U_N)$ using the symmetry approach. In particular, we classify second-order and third-order systems that are polynomial in u, U and their derivatives. This work was initiated by Vladimir Sokolov and the second author (TW) in [34]. Here, N is an arbitrary positive integer and the product between two vectors is defined by the scalar product

$$\left\langle \partial_x^m U, \partial_x^n U \right\rangle \equiv \sum_{j=1}^{N} \left(\partial_x^m U_j\right)\left(\partial_x^n U_j\right), \qquad m, n \geqslant 0.$$

We do not consider constant vectors C_j or matrices C_{jk} as in $\sum_{j=1}^{N} C_j \left(\partial_x^m U_j\right)$ or, for example, $\sum_{j,k=1}^{N} C_{jk}\left(\partial_x^m U_j\right)\left(\partial_x^n U_k\right)$. Moreover, we require that the scalar and vector evolution equations are *truly* coupled, that is, U occurs in $u_t = \cdots$ and u occurs in $U_t = \cdots$. Classifications described in this paper are restricted to (λ_1, λ_2)-*homogeneous* systems of *weight* μ. These are systems that admit the one-parameter group of scaling symmetries

$$(x, t, u, U_j) \longrightarrow (a^{-1}x, a^{-\mu}t, a^{\lambda_1}u, a^{\lambda_2}U_j), \qquad a \neq 0.$$

[3] Here we mean conservation laws that do not depend on x and t explicitly.

[4] The list given in [16] seems to be incomplete, because we cannot identify an integrable system of the Burgers type (cf (4.3) in this paper) with any system in the list.

We consider only systems with $\lambda_1, \lambda_2 > 0$ and a differential order equal to μ. For systems with $\lambda_1 = \lambda_2$, this would imply the existence of a linear leading part (dispersion), but not in the case of mixed systems with $\lambda_1 \neq \lambda_2$. For example, for $\mu = 2$ and $\lambda_1 = 2\lambda_2$, the two terms u_{xx} and $\langle U, U_{xx} \rangle$ have the same weight and a differential order equal to μ. In either case, systems having a degenerate leading part are also included in our classification.

For the scalar case, it was proven in [35] that a λ-homogeneous polynomial evolution equation with $\lambda > 0$ and a dispersion term may possess a polynomial higher symmetry only if

$\lambda = 2$ (KdV weighting),
$\lambda = 1$ (Burgers/pKdV/mKdV weighting) or
$\lambda = \frac{1}{2}$ (Ibragimov–Shabat weighting [36]).

It was also proven in [35] that any symmetry-integrable[5] equation of second (third) order in the considered classes *does* possess a symmetry of third (fifth) order, respectively. Similar results on (λ_1, λ_2)-homogeneous polynomial systems of weight 2 with two components were obtained in [14]. Under the conditions of $\lambda_1, \lambda_2 > 0$, $|\lambda_1 - \lambda_2| \notin \mathbb{N}_{>0}$, a nondegeneracy of the linear part[6] and the order of the nonlinear part less than 2, such a system may possess polynomial higher symmetries only if $\lambda_1 = \lambda_2 = 2, 1, \frac{1}{2}$ or

$$\lambda_1 = \tfrac{1}{3}, \lambda_2 = \tfrac{2}{3};$$
$$\lambda_1 = \tfrac{2}{3}, \lambda_2 = \tfrac{1}{3}.$$

In these classes, any symmetry-integrable system of second order *does* possess a symmetry of third order or fourth order. Since we study $(1 + N)$-component systems of second and third order that may have a degenerate leading part, we cannot entirely rely on these results. They neither give all possible pairs of (λ_1, λ_2) for integrable cases nor indicate the order of a higher symmetry to exist. Nevertheless, in this paper, we concentrate our attention on systems that are homogeneous in $(\lambda_1, \lambda_2) = (2, 2), (1, 1), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{3}, \frac{2}{3}\right)$ or $\left(\frac{2}{3}, \frac{1}{3}\right)$ and $\mu = 2$ or 3. Indeed, as we will see below, there exist a lot of interesting integrable systems in these classes. Classifications for other pairs of (λ_1, λ_2) will be reported in a subsequent paper.

The search for integrable systems in this paper is based on the simplest version of the symmetry approach [1, 2], i.e. the existence of one higher symmetry. It is considered as a necessary, but in general not sufficient condition for integrability[7]. Both 'S-integrable' and 'C-integrable' systems can be detected by the existence of one higher symmetry. To do concrete computations using the computer algebra program Crack [40], we assume the existence of a third-order or a fourth-order symmetry for a second-order system and a fifth-order symmetry for a third-order system. Although the existence of symmetries of a specific order may be too restrictive and not necessary for integrability[8], it allows us to perform exhaustive searches and obtain complete lists for these cases. An overview of the performed computations is given in the next section. For the lists generated by computer, we first remove inessential parameters by scaling independent and dependent variables. We note that a linear change of dependent variables mixing the scalar u with components of the vector U would give systems with more than one scalar unknown and is therefore not considered. Next, we prove

[5] The symmetry-integrable equations are such equations that possess an infinite set of (commuting) higher symmetries.

[6] For full details, see [14].

[7] Some systems of the Bakirov type are known to possess only a finite number of higher symmetries [37–39]. However, all such examples are pathological and less interesting, because they are already in their given form triangular linear. In this paper, we encounter systems with a higher symmetry that are reducible to a triangular form by a *nonlinear* and *non-ultralocal* change of variables. It is an open question whether such systems are symmetry-integrable in general.

[8] As a result, we may miss some integrable cases.

integrability for nearly all listed systems by constructing either a Lax representation or a linearizing transformation[9]. Some systems in the lists can be reduced to triangular systems by a nonlinear transformation of dependent variables. If that is possible then systems contain a closed subsystem in a nontrivial manner. In that case, we first prove the integrability of the subsystem and then discuss how to solve the remaining equations. We can reduce the task of proving the integrability through establishing relationships among the listed systems. We construct a rich set of Miura-type transformations, including Miura maps plus potentiation, that connect different systems in the lists. Thus, we have only to investigate one representative for each group of connected systems.

This paper is organized as follows. In section 2, we explain briefly how the lists of systems in this paper are generated by computer. In section 3, we perform a classification of second-order and third-order systems in the $\lambda_1 = \lambda_2 = 2$ (KdV weighting) case. The list of second-order systems with a third-order or a fourth-order symmetry is empty, while that of third-order systems with a fifth-order symmetry consists of four members. The list itself is already known [34] but we prove their integrability in section 3.

Section 4 forms the *main part* of this paper. We classify second-order and third-order systems in the $\lambda_1 = \lambda_2 = 1$ (Burgers/pKdV/mKdV weighting) case. The list of second-order systems consists of three members that generalize the Burgers equation. All these systems possess both a third-order and a fourth-order symmetry. We can (triangular) linearize two of them through an extension of the Hopf–Cole transformation, while integrability[10] of the other system, (4.5), *remains unproven*. We discuss travelling-wave solutions of (a subsystem of) this system to indicate its nontrivial nature. The interested reader is referred to section 4.2.3 for the details. The list of third-order systems consists of 25 members, three of which are symmetries of the second-order systems from the previous list. Consequently, we can (triangular) linearize two of the three third-order systems, while integrability of the other third-order system remains to be seen. Another third-order system in the list (system (4.9)) is very close to the latter system and we do not know how to integrate it either. For the other 21 (= 25 − 3 − 1) systems of third order, we prove that they are integrable or, at least, they contain an integrable closed subsystem. We point out that one of the 21 systems is the third-order symmetry of a nontrivial first-order system. Miura-type transformations that connect third-order systems in the lists of $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = \lambda_2 = 2$ are presented.

In section 5, we classify second-order and third-order systems in the $\lambda_1 = \lambda_2 = \frac{1}{2}$ (Ibragimov–Shabat weighting) case. The list of second-order systems is empty, while that of third-order systems consists of two members. We can linearize both of them through a generalization of the linearizing transformation for the Ibragimov–Shabat equation. In section 6, we obtain negative results regarding a classification in the case of $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$. In section 7, we perform a classification of second-order and third-order systems in the case of $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3}$. We obtain one second-order system with a third-order symmetry, two second-order systems with a fourth-order symmetry and two third-order systems with a fifth-order symmetry. Thereby we have one second-order system without a third-order symmetry, but with a fourth-order symmetry. All the listed systems can be linearized through an ultralocal change of dependent variables. Section 8 is devoted to concluding remarks.

Finally, we would like to mention that our results in section 4 refine and generalize the recent work of Foursov and Olver [18, 25, 26]. Their work focused on polynomial systems of two symmetrically coupled nonlinear evolution equations, i.e. symmetric systems for two scalar unknowns. They obtained the complete lists of $\lambda_1 = \lambda_2 = 1$ homogeneous systems of

[9] In this paper, we are not going to pursue the symmetry integrability.

[10] We mean the existence of either a Lax representation or a linearizing transformation.

second and third order with two higher symmetries of specific orders. Most of the $(1 + N)$-component systems listed in section 4 generalize two-component systems of Foursov–Olver, up to a linear change of dependent variables. To see this, we remark that, because of our assumption on the admissible multiplications, the evolution equation for the scalar u is even in the vector U, while the equation for U is odd in U. Therefore, we can symmetrize our systems in the special case $N = 1$ through the linear change of variables: $u = a(q+r)$, $U = b(q-r)$, where a and b are nonzero constants.

On the basis of this re-formulation, we compare in section 4 our lists with those of Foursov–Olver. A brief summary of the comparison results is as follows:

- Any system in Foursov–Olver's lists[11] corresponds to the $N = 1$ case of one or two systems in our lists. This means that, after the linear change of dependent variables mentioned above, their two-component systems always admit $(1 + N)$-component generalization(s) preserving the integrability. This result is quite interesting, but unlikely to hold true in general for other classes of two-component systems.
- Some systems in our lists do not have any counterpart in Foursov–Olver's lists. They are systems that become the trivial equation $u_t = 0$ under the reduction $U = \mathbf{0}$. Such systems were excluded from consideration in the work of Foursov–Olver by their assumption on strong nondegeneracy of the linear part (see e.g. section 2 of [18]). In this respect, our lists are richer than Foursov–Olver's lists even in the $N = 1$ case.

Besides extending Foursov–Olver's lists [18, 25, 26], we prove the integrability of many systems in their lists for the first time. We also correct errors in [18, 25, 26] and point out overlooked references in which some systems in their lists were studied earlier.

2. Computational aspects

Before describing the classification results in sections 3–7, in this section, we would like to make some comments on the computations performed.

As the first step, a homogeneous ansatz for a system consisting of a scalar equation $u_t = \cdots$ and a vector equation $U_t = \cdots$ is generated together with a system of higher symmetry equations $u_\tau = \cdots$, $U_\tau = \cdots$. Each term has a different undetermined coefficient. We assume that these coefficients do not depend on N (the number of components of U), and that N is not fixed at any specific value[12].

Computationally more expensive is the formulation of the symmetry conditions $u_{[t,\tau]} = 0$, $U_{[t,\tau]} = \mathbf{0}$. For low values of λ_1, λ_2 and high differential order the right-hand sides of the system and the symmetry involve many terms, and in addition each of the terms has an increasing number of factors. Higher order x-derivatives of such terms cause a large expression swell, too large to compute the commutators in one step. We therefore perform the computation of $u_{[t,\tau]}$ and $U_{[t,\tau]}$ in stages. Because the right-hand sides of the system and the symmetry do not involve $\partial_t, \partial_\tau$, substitutions of u_t, U_t, u_τ, U_τ in the commutators are done only once. Consequently, commutators are linear in the coefficients of the system and coefficients of the symmetry. To exploit this linearity, we partition

$$\begin{aligned} u_t &= \sum_i F_i, & U_t &= \sum_i G_i, & u_\tau &= \sum_i H_i, \\ U_\tau &= \sum_i K_i, & u_{[t,\tau]} &= \sum_i P_i, & U_{[t,\tau]} &= \sum_i Q_i, \end{aligned} \tag{2.1}$$

[11] We mean the lists of systems that are not reducible to a triangular form by a linear change of dependent variables.
[12] The arbitrariness of N is crucial for functional independence of the scalar products $\langle \partial_x^m U, \partial_x^n U\rangle$ $(0 \leqslant m \leqslant n)$ [34].

where the expressions $F_i, G_i, H_i, K_i, P_i, Q_i$ contain only terms with a total degree i of all scalar vector products of U and x-derivatives of U (for example, $\langle U, U_x\rangle U$ having degree 1). By using the observation that the number of scalar vector products in a term does not change when a term is differentiated, we can compute each P_i independently through

$$P_i = \sum_{j=0}^{i} u_{[t,\tau]}|_{u_t=F_j,\, U_t=G_j,\, u_\tau=H_{i-j},\, U_\tau=K_{i-j}},$$

and similarly for each Q_i. Because they are the only terms that have ith degree powers of scalar vector products, all P_i, Q_i must vanish identically. After one single P_i or Q_i is computed, it can be split[13] and some of the consequences, such as the vanishing of some coefficients, can be used to simplify F_i, G_i, H_i, K_i before computing the next P_j and Q_j.

For large problems (low λ and high differential order), the computation of Q_i was still too memory intensive[14] so that another partitioning of the computation was implemented. In this level of partitioning, first those terms in F_i, G_i, H_i, K_i which can contribute to the highest derivative vectorial factor $\partial_x^j U$ in Q_i were considered. Let us call the partial commutator that comes out of this computation $\hat{C}_{i,j}$. From $\hat{C}_{i,j}$ all the terms with vectorial factor $\partial_x^j U$ are extracted, split, and the remaining terms from $\hat{C}_{i,j}$ (with a vectorial factor of order $<j$) are carried over for the computation of the next terms with derivative vectorial factor $\partial_x^{j-1} U$ in Q_i.[15]

In this way, the computation of single large expressions $u_{[t,\tau]}, U_{[t,\tau]}$ is avoided and replaced by the computation of many partial commutators resulting in bilinear algebraic equations for the undetermined coefficients.

To each list of conditions is attached a list of inequalities which have to be fulfilled by any solution. A first inequality results from the requirement that at least one of both equations involves at least one x-derivative of the required order (second or third). Similarly, the symmetry equations have to involve at least one x-derivative of the required order and the right-hand sides of both symmetry equations must be nonzero. Two further conditions prevent the generation of triangular integrable systems by requiring that U occurs in $u_t = \cdots$ and u occurs in $U_t = \cdots$.

The solution of the overdetermined bilinear algebraic systems was accomplished with the computer program CRACK written for the solution of overdetermined algebraic but also differential systems. One technique that proved to be quite useful in general, especially for the solution of larger systems with $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = \lambda_2 = \frac{1}{2}$, is an equation shortening method described in [41].

Tables 1–3 give an overview on the complexity of computations. These have been performed on a 1.7 GHz Pentium 4 PC running the computer algebra system REDUCE 3.7 in a 120 MB session under Linux. Quoted execution times are sensitive to settings of computing parameters and should be taken only as rough indicators.

For $\lambda_1 = \lambda_2 = 1$ and orders 3 + 5, the large number of solutions has the consequence that the system of algebraic conditions does not simplify so readily and is more complicated to solve. Hence, the program has more often to impose case distinctions where an unknown is assumed to be at first zero and then nonzero. As a result, solutions may be found which can be unified into a single solution. This is the case if, for example, one solution S_1 includes the condition $a_{17} = 0$ while the other solution S_2 requires $a_{17} \neq 0$ for some undetermined

[13] By *splitting* we mean the extraction and setting to zero of all coefficients of all products of all powers of all scalar and vector functions and their derivatives.

[14] The computation was too memory demanding to be performed on the computers available to one of the authors in 2000.

[15] More details can be obtained at request.

Table 1. Computations in the orders 2 + 3 problem for the five weightings.

λ_1, λ_2	2, 2	1, 1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{2}{3}, \frac{1}{3}$
Number of unknowns in the system	5	10	15	10	13
Number of unknowns in the symmetry	6	21	36	24	22
Total number of unknowns	11	31	51	34	35
Number of conditions	13	66	149	102	114
Total number of terms in all conditions	34	341	1093	529	694
Average number of terms in a condition	2.6	5.2	7.3	5.2	6.1
Time to formulate algebraic conditions	0.5 s	1.8 s	8 s	3.2 s	6.3 s
Time to solve conditions	0.5 s	29 s	29 s	45 s	22 s
Number of solutions	0	3	0	0	1

Table 2. Computations in the orders 2 + 4 problem for the five weightings.

λ_1, λ_2	2, 2	1, 1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{2}{3}, \frac{1}{3}$
Number of unknowns in the system	5	10	15	10	13
Number of unknowns in the symmetry	12	39	79	54	66
Total number of unknowns	17	49	94	64	79
Number of conditions	26	123	313	215	276
Total number of terms in all conditions	77	770	3096	1462	2435
Average number of terms in a condition	3.0	6.3	9.9	6.8	8.8
Time to formulate algebraic conditions	1 s	5 s	48 s	13 s	48 s
Time to solve conditions	0.4 s	1 min 58 s	3 min 44 s	1 min 23 s	3 min 40 s
Number of solutions	0	3 [a]	0	0	2

[a] Although the program CRACK originally produced four solutions, we could easily recognize that one solution is a special case of another.

Table 3. Computations in the orders 3 + 5 problem for the five weightings.

λ_1, λ_2	2, 2	1, 1	$\frac{1}{2}, \frac{1}{2}$	$\frac{1}{3}, \frac{2}{3}$	$\frac{2}{3}, \frac{1}{3}$
Number of unknowns in the system	6	21	36	24	22
Number of unknowns in the symmetry	17	74	164	115	126
Total number of unknowns	23	95	200	139	148
Number of conditions	50	386	1154	798	955
Total number of terms in all conditions	218	5000	27 695	12 694	17 385
Average number of terms in a condition	4.4	13	24	16	18
Time to formulate algebraic conditions	5 s	2 min 52 s	2 h 7 min	23 min 45 s	41 min 18 s
Time to solve conditions	6.5 s	5 h 47 min	1 day [a]	1 h 20 min	1 h 7 min
Number of solutions	4	25	2	0	2

[a] The computation involved one manual interference.

coefficient a_{17}, and if setting $a_{17} = 0$ in S_2 makes both solutions equivalent, in the system and in the symmetry. Sometimes a substitution like $a_{17} = 0$ in S_2 may cause a division by zero which can be avoided by re-parametrizing S_2. An algorithm and its implementation in a computer program analysing such situations have recently been developed and applied.

On the Web page http://lie.math.brocku.ca/twolf/htdocs/sv/over.html, one can inspect the original systems of conditions and the solutions as well as download them in machine-readable form. In addition to investigating systems of differential orders $2+3$ (for system + symmetry), $2+4$ and $3+5$, we also investigated orders $1+2$ and $1+3$. The main purpose was to recognize whether a second-order or a third-order system is actually the symmetry of a nontrivial first-order system. Details can also be found on the above-mentioned Web page. The package Crack can be obtained from http://lie.math.brocku.ca/twolf/crack/.

After all solutions have been determined, the task of proving integrability follows. In the process of identifying and classifying some of the constant coefficient systems, the *Mathematica* package 'InvariantsSymmetries.m' [42] has been used to compute conservation laws and higher symmetries.

3. The case $\lambda_1 = \lambda_2 = 2$: coupled KdV equations

In this section, we classify second-order and third-order systems in the $\lambda_1 = \lambda_2 = 2$ (KdV weighting) case. In the first part (section 3.1), we present a complete list of such systems with a specific order symmetry (the list is already known, see [34]). In the second part (section 3.2), we prove the integrability of the listed systems.

3.1. List of systems with a higher symmetry

The general ansatz for a $\lambda_1 = \lambda_2 = 2$ homogeneous evolutionary system of second order for a scalar function u and a vector function U takes the form

$$\begin{cases} u_{t_2} = a_1 u_{xx} + a_2 u^2 + a_3 \langle U, U\rangle, \\ U_{t_2} = a_4 U_{xx} + a_5 u U. \end{cases} \tag{3.1}$$

The following constraints guarantee the order to be 2 and the system not to be triangular:

$$(a_1, a_4) \neq (0, 0), \qquad a_3 \neq 0, \qquad a_5 \neq 0.$$

Similarly, the general ansatz for a third-order system takes the form

$$\begin{cases} u_{t_3} = b_1 u_{xxx} + b_2 u u_x + b_3 \langle U, U_x\rangle, \\ U_{t_3} = b_4 U_{xxx} + b_5 u_x U + b_6 u U_x, \end{cases} \tag{3.2}$$

for which the following constraints guarantee the order to be 3 and the system not to be triangular:

$$(b_1, b_4) \neq (0, 0), \qquad b_3 \neq 0, \qquad (b_5, b_6) \neq (0, 0).$$

However, we relax these constraints as

$$(b_1, b_4) \neq (0, 0), \qquad (b_1, b_2, b_3) \neq (0, 0, 0), \qquad (b_4, b_5, b_6) \neq (0, 0, 0),$$

when we consider a third-order symmetry for a second-order system, as stated in section 2. We omit the general ansatz for a fourth-order or a fifth-order system here (in the $\lambda_1 = \lambda_2 = 2$ case) and hereafter (in the other weightings) because of its increased length. However, it is available on the above-mentioned Internet site.

Proposition 3.1. *No second-order system of the form (3.1) with a third-order symmetry of the form (3.2) or a fourth-order symmetry exists.*

Theorem 3.2. *Any third-order system of the form (3.2) with a fifth-order symmetry has to coincide with one of the following four systems up to a scaling of t_3, x, u, U (we omit the subscript of t_3):*

$$\begin{cases} u_t = \langle U, U_x\rangle, \\ U_t = U_{xxx} + u_x U + 2u U_x, \end{cases} \tag{3.3}$$

$$\begin{cases} u_t = u_{xxx} + 6uu_x - 6\langle U, U_x \rangle, \\ U_t = U_{xxx} + 6u_x U + 6uU_x, \end{cases} \tag{3.4}$$

$$\begin{cases} u_t = u_{xxx} + 3uu_x + 3\langle U, U_x \rangle, \\ U_t = u_x U + uU_x, \end{cases} \tag{3.5}$$

$$\begin{cases} u_t = u_{xxx} + 6uu_x - 12\langle U, U_x \rangle, \\ U_t = -2U_{xxx} - 6uU_x. \end{cases} \tag{3.6}$$

All systems (3.3)–(3.6) admit the reduction $U = \mathbf{0}$. From this viewpoint, (3.3) is a generalization of the trivial equation $u_t = 0$, while (3.4)–(3.6) are generalizations of the KdV equation.

3.2. *Integrability of systems (3.3)–(3.6)*

3.2.1. System (3.3). System (3.3) is a multi-component generalization of one of the Drinfel'd–Sokolov systems [43, 44]. The integrability of this system has been established in the literature [45–47].

3.2.2. System (3.4). System (3.4) is known as a Jordan KdV system [27, 28, 34, 48]. Let us briefly summarize its integrability. It is well known that the matrix KdV equation,

$$Q_t = Q_{xxx} + 3(Q^2)_x, \tag{3.7}$$

admits a Lax representation [49–52]. Then, system (3.4) is also integrable, because it is obtained from (3.7) through the following reduction:

$$Q = u\mathbf{1} + \sum_{j=1}^{N} U_j \mathbf{e}_j.$$

Here $\mathbf{1}$ is the identity matrix and $\{\mathbf{e}_1, \ldots, \mathbf{e}_N\}$ are mutually anti-commuting matrices that satisfy the condition

$$\{\mathbf{e}_i, \mathbf{e}_j\}_+ \equiv \mathbf{e}_i \mathbf{e}_j + \mathbf{e}_j \mathbf{e}_i = -2\delta_{ij}\mathbf{1}.$$

3.2.3. System (3.5). System (3.5) is a multi-component generalization of the Zakharov–Ito system [52, 53] and corresponds to a special case of the coupled KdV equations considered by Kupershmidt [54]. Introducing a new variable w by $w \equiv \sqrt{\langle U, U \rangle}$, we find that (3.5) contains the original Zakharov–Ito system

$$\begin{cases} u_t = u_{xxx} + 3uu_x + 3ww_x, \\ w_t = (uw)_x. \end{cases} \tag{3.8}$$

Therefore, system (3.5) is a triangular system that consists of the Zakharov–Ito system and the linear equation for U with Zakharov–Ito-system-dependent coefficients.

To demonstrate the integrability of the whole system (3.5), we first summarize a Lax representation for the Zakharov–Ito system [52, 55, 56]. We consider a pair of linear equations for a scalar function ψ,

$$\begin{cases} \psi_{xx} = (\zeta + q + \zeta^{-1} r)\psi, \\ \psi_t = (4\zeta - 2q)\psi_x + q_x \psi, \end{cases} \tag{3.9}$$

where ζ is the spectral parameter. Then, the compatibility condition $\psi_{xxt} = \psi_{txx}$ for (3.9) implies the following system:

$$\begin{cases} q_t = q_{xxx} - 6qq_x + 4r_x, \\ r_t = -4q_x r - 2qr_x. \end{cases}$$

This system coincides with the Zakharov–Ito system (3.8) through the change of dependent variables, $q = -u/2,\ r = -3w^2/16$. It is noteworthy that the quantity $1/\psi^2$ in the limit $\zeta \to 0$ obeys the same evolution equation as that for U, namely $U_t = (uU)_x$.

Next, we fix a solution of subsystem (3.8) and discuss solutions of the linear equation for U. For the sake of simplicity, we assume that $w(x,t)$ in the fixed solution is not a trivial function. Then, noting the relation $w_t = (uw)_x$, we obtain the following solution to the equation for U:

$$U_j = w \cdot f_j\left(\int^x w\,\mathrm{d}x'\right), \qquad j = 1, 2, \ldots, N.$$

Here $f_1(z), \ldots, f_N(z)$ are arbitrary functions of z, except that they must satisfy one constraint, $\sum_{j=1}^N [f_j(z)]^2 = 1$, due to the relation $\langle U, U\rangle = w^2$. For the case in which $w(x,t)$ is identically zero, we mention some references in section 4.2.9.

3.2.4. System (3.6). System (3.6) is a multi-component generalization[16] of the two-component KdV system ((3.6) with $N = 1$) proposed by Hirota and Satsuma [57]. Actually, the Hirota–Satsuma system is also understood as an example of the Kac–Moody KdV systems studied independently by Drinfel'd and Sokolov [43, 44]. A Lax representation for the Hirota–Satsuma system was constructed in [59][17]. Recently, it was generalized to the three-component case ((3.6) with $N = 2$) by Wu *et al* [61]. Let us demonstrate that (3.6) admits a Lax representation in the general case of N-component vector U. We consider a set of linear equations for two column-vector functions ψ and ϕ,

$$\begin{cases} \psi_{xx} + P\psi + Q\phi = \zeta\psi, \\ \phi_{xx} + P\phi + R\psi = -\zeta\phi, \\ \psi_t = 4\zeta\psi_x + 2P\psi_x - 4Q\phi_x - P_x\psi + 2Q_x\phi, \\ \phi_t = -4\zeta\phi_x + 2P\phi_x - 4R\psi_x - P_x\phi + 2R_x\psi. \end{cases}$$

Here, ζ is the spectral parameter and P, Q and R are square matrices with the same dimension. The compatibility conditions $\psi_{xxt} = \psi_{txx},\ \phi_{xxt} = \phi_{txx}$ imply a system of three matrix equations

$$\begin{cases} P_t = P_{xxx} + 3(P^2)_x - 6(QR)_x, \\ Q_t = -2Q_{xxx} - 6Q_x P + 3[P_x, Q], \\ R_t = -2R_{xxx} - 6R_x P + 3[P_x, R], \end{cases} \tag{3.10}$$

together with three constraints

$$[P, Q] = O, \qquad [P, R] = O, \qquad [Q, R]_x = O.$$

If we consider the reduction

$$P = u\mathbf{1}, \qquad Q = U_1\mathbf{1} + \sum_{j=1}^{N-1} U_{j+1}\mathbf{e}_j, \qquad R = U_1\mathbf{1} - \sum_{j=1}^{N-1} U_{j+1}\mathbf{e}_j, \qquad \{\mathbf{e}_i, \mathbf{e}_j\}_+ = -2\delta_{ij}\mathbf{1},$$

the three constraints are automatically satisfied and system (3.10) is reduced to the multi-component Hirota–Satsuma system (3.6).

[16] This multi-component generalization was proposed in [58], but the integrability was not discussed in that paper.
[17] Vladimir Sokolov commented that the Lax representation was reported earlier in the Russian paper [60], which is not accessible to the authors.

4. The case $\lambda_1 = \lambda_2 = 1$: coupled Burgers, pKdV and mKdV equations

In this section, we classify second-order and third-order systems in the $\lambda_1 = \lambda_2 = 1$ (Burgers/pKdV/mKdV weighting) case. In the first part (section 4.1), we present complete lists of such systems with a specific order symmetry. In the second part (section 4.2), we discuss the integrability of the listed systems and compare them with Foursov–Olver's two-component systems [18, 25, 26] through symmetrization as stated in the introduction.

4.1. Lists of systems with a higher symmetry

The general ansatz for a $\lambda_1 = \lambda_2 = 1$ homogeneous evolutionary system of second order for a scalar function u and a vector function U takes the form

$$\begin{cases} u_{t_2} = a_1 u_{xx} + a_2 u u_x + a_3 u^3 + a_4 u\langle U, U\rangle + a_5\langle U, U_x\rangle, \\ U_{t_2} = a_6 U_{xx} + a_7 u_x U + a_8 u U_x + a_9 u^2 U + a_{10}\langle U, U\rangle U. \end{cases} \tag{4.1}$$

The following constraints guarantee the order to be 2 and the system not to be triangular:

$$(a_1, a_6) \neq (0, 0), \qquad (a_4, a_5) \neq (0, 0), \qquad (a_7, a_8, a_9) \neq (0, 0, 0).$$

Similarly, the general ansatz for a third-order system takes the form

$$\begin{cases} u_{t_3} = b_1 u_{xxx} + b_2 u u_{xx} + b_3 u_x^2 + b_4 u^2 u_x + b_5 u^4 + b_6 u_x\langle U, U\rangle \\ \qquad + b_7 u\langle U, U_x\rangle + b_8\langle U, U_{xx}\rangle + b_9\langle U_x, U_x\rangle + b_{10} u^2\langle U, U\rangle + b_{11}\langle U, U\rangle^2, \\ U_{t_3} = b_{12} U_{xxx} + b_{13} u_{xx} U + b_{14} u_x U_x + b_{15} u U_{xx} + b_{16} u u_x U \\ \qquad + b_{17} u^2 U_x + b_{18}\langle U, U\rangle U_x + b_{19}\langle U, U_x\rangle U + b_{20} u^3 U + b_{21} u\langle U, U\rangle U, \end{cases} \tag{4.2}$$

for which the following constraints guarantee the order to be 3 and the system not to be triangular: $(b_1, b_{12}) \neq (0, 0)$ and at least one of $b_6, \ldots, b_{11}$ and one of $b_{13}, \ldots, b_{17}, b_{20}, b_{21}$ must not vanish. However, when we consider a third-order symmetry for a second-order system, we relax these constraints as follows (cf section 2): $(b_1, b_{12}) \neq (0, 0)$ and at least one of $b_1, \ldots, b_{11}$ and one of $b_{12}, \ldots, b_{21}$ must not vanish.

Theorem 4.1. *Any second-order system of the form (4.1) with a third-order symmetry of the form (4.2) has to coincide with one of the following three systems up to a scaling of t_2, x, u, U (we omit the subscript of t_2):*

$$\begin{cases} u_t = \frac{1}{3}(1 + 2a)(u_{xx} + 2uu_x) + \frac{4}{3}\langle U, U_x\rangle, \\ U_t = U_{xx} + \frac{1}{3}(1 - a)u_x U + uU_x + \frac{1}{12}(1 - 4a)u^2 U - \frac{1}{3}\langle U, U\rangle U, \\ \qquad a \text{ is arbitrary}, \end{cases} \tag{4.3}$$

$$\begin{cases} u_t = u_{xx} + 2uu_x + 2\langle U, U_x\rangle, \\ U_t = -\frac{1}{2}u_x U - \frac{1}{2}u^2 U - \frac{1}{2}\langle U, U\rangle U, \end{cases} \tag{4.4}$$

$$\begin{cases} u_t = u_{xx} + 2uu_x + \langle U, U_x\rangle, \\ U_t = \frac{1}{2}u_x U + uU_x. \end{cases} \tag{4.5}$$

Proposition 4.2. *Any second-order system of the form (4.1) with a fourth-order symmetry has to coincide with one of the three systems (4.3)–(4.5) up to a scaling of t_2, x, u, U.*

All systems (4.3)–(4.5) admit the reduction $U = \mathbf{0}$. From this viewpoint, (4.3)–(4.5) are considered as generalizations of the Burgers equation.

Theorem 4.3. *Any third-order system of the form (4.2) with a fifth-order symmetry has to coincide with one of the following 25 systems up to a scaling of t_3, x, u, U (we omit the subscript of t_3):*

$$\begin{cases} u_t = a\big(u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x\big) + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \qquad + 2\langle U, U_{xx}\rangle + 2\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + \frac{1}{2}(1-a)u_{xx}U + \frac{3}{2}u_xU_x + \frac{3}{2}uU_{xx} + \frac{3}{4}(1-2a)uu_xU \\ \qquad + \frac{3}{4}u^2U_x - \langle U, U_x\rangle U + \frac{1}{8}(1-4a)u^3U - \frac{1}{2}u\langle U, U\rangle U, \qquad a \text{ is arbitrary}, \end{cases} \tag{4.6}$$

$$\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \qquad + 2\langle U, U_{xx}\rangle + 2\langle U_x, U_x\rangle, \\ U_t = -\frac{1}{2}u_{xx}U - \frac{3}{2}uu_xU - \langle U, U_x\rangle U - \frac{1}{2}u^3U - \frac{1}{2}u\langle U, U\rangle U, \end{cases} \tag{4.7}$$

$$\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \qquad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = \frac{1}{2}u_{xx}U + u_xU_x + uu_xU + u^2U_x + \frac{1}{2}\langle U, U\rangle U_x + \frac{1}{2}\langle U, U_x\rangle U, \end{cases} \tag{4.8}$$

$$\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \qquad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = \frac{1}{2}u_{xx}U + u_xU_x + uu_xU + u^2U_x + \langle U, U\rangle U_x, \end{cases} \tag{4.9}$$

$$\begin{cases} u_t = 3u_x\langle U, U\rangle + 3\langle U, U_{xx}\rangle - 3\langle U, U\rangle^2, \\ U_t = U_{xxx} + u_{xx}U + u_xU_x - 3\langle U, U_x\rangle U, \end{cases} \tag{4.10}$$

$$\begin{cases} u_t = 2u_x\langle U, U\rangle + 2\langle U, U_{xx}\rangle - \langle U_x, U_x\rangle - 2\langle U, U\rangle^2, \\ U_t = U_{xxx} + u_{xx}U + 2u_xU_x - 2\langle U, U\rangle U_x - 2\langle U, U_x\rangle U, \end{cases} \tag{4.11}$$

$$\begin{cases} u_t = u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + u_{xx}U + u_xU_x - 2uu_xU - u^2U_x + \langle U, U\rangle U_x - \langle U, U_x\rangle U, \end{cases} \tag{4.12}$$

$$\begin{cases} u_t = u_{xxx} + \frac{3}{2}u_x^2 + \frac{3}{2}\langle U_x, U_x\rangle, \\ U_t = u_xU_x, \end{cases} \tag{4.13}$$

$$\begin{cases} u_t = u_{xxx} + 3u_x^2 + 2au_x\langle U, U\rangle + a\langle U, U_{xx}\rangle + a\langle U_x, U_x\rangle + b\langle U, U\rangle^2, \\ U_t = u_{xx}U + 2u_xU_x + a\langle U, U\rangle U_x + a\langle U, U_x\rangle U, \qquad (a, b) \neq (0, 0), \end{cases} \tag{4.14}$$

$$\begin{cases} u_t = u_{xxx} + 3u_x^2 - 3\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_xU_x, \end{cases} \tag{4.15}$$

$$\begin{cases} u_t = u_{xxx} + 3u_x^2 + u_x\langle U, U\rangle + \langle U, U_{xx}\rangle, \\ U_t = U_{xxx} + 3u_{xx}U + 3u_xU_x + \langle U, U_x\rangle U, \end{cases} \tag{4.16}$$

$$\begin{cases} u_t = u_{xxx} + 3u_x^2 + 2u_x\langle U, U\rangle + \langle U, U_{xx}\rangle + \frac{1}{2}\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_{xx}U + 6u_xU_x + 2\langle U, U_x\rangle U, \end{cases} \tag{4.17}$$

$$\begin{cases} u_t = u_{xxx} + 3u_x^2 + 4u_x\langle U, U\rangle + 2\langle U, U_{xx}\rangle + \langle U_x, U_x\rangle + \frac{2}{3}\langle U, U\rangle^2, \\ U_t = -2U_{xxx} - 6u_{xx}U - 6u_xU_x - 4\langle U, U_x\rangle U, \end{cases} \tag{4.18}$$

$$\begin{cases} u_t = u_{xxx} + u_x^2 - 12\langle U, U_{xx}\rangle + 12\langle U_x, U_x\rangle - 4\langle U, U\rangle^2, \\ U_t = 4U_{xxx} + u_{xx}U + 2u_xU_x + 4\langle U, U\rangle U_x + 4\langle U, U_x\rangle U, \end{cases} \tag{4.19}$$

$$\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x + \frac{3}{2}\langle U, U\rangle U_x, \end{cases} \tag{4.20}$$

$$\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x + \frac{1}{2}\langle U, U\rangle U_x + \langle U, U_x\rangle U, \end{cases} \tag{4.21}$$

$$\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{1}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = u_{xx}U + u_xU_x - uu_xU - \frac{1}{2}u^2U_x + \frac{1}{2}\langle U, U\rangle U_x + \langle U, U_x\rangle U, \end{cases} \tag{4.22}$$

$$\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle + \frac{1}{2}\langle U, U\rangle^2, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x - \frac{1}{2}\langle U, U\rangle U_x + \frac{1}{2}u\langle U, U\rangle U, \end{cases} \tag{4.23}$$

$$\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle \\ \qquad - \frac{1}{4}u^2\langle U, U\rangle + \frac{1}{4}\langle U, U\rangle^2, \\ U_t = \frac{1}{2}u_{xx}U + \frac{1}{2}\langle U, U_x\rangle U - \frac{1}{4}u^3U + \frac{1}{4}u\langle U, U\rangle U, \end{cases} \tag{4.24}$$

$$\begin{cases} u_t = u_{xxx} + u^2u_x + u_x\langle U, U\rangle, \\ U_t = U_{xxx} + u^2U_x + \langle U, U\rangle U_x, \end{cases} \tag{4.25}$$

$$\begin{cases} u_t = u_{xxx} + 2u^2u_x + u_x\langle U, U\rangle + u\langle U, U_x\rangle, \\ U_t = U_{xxx} + uu_xU + u^2U_x + \langle U, U\rangle U_x + \langle U, U_x\rangle U, \end{cases} \tag{4.26}$$

$$\begin{cases} u_t = u_{xxx} - 6u^2u_x + 6u_x\langle U, U\rangle + 12u\langle U, U_x\rangle, \\ U_t = U_{xxx} - 12uu_xU - 6u^2U_x + 6\langle U, U\rangle U_x, \end{cases} \tag{4.27}$$

$$\begin{cases} u_t = u_{xxx} - 6u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 3u_{xx}U + 3u_xU_x - 6uu_xU - 3u^2U_x + \langle U, U\rangle U_x + 3\langle U, U_x\rangle U, \end{cases} \tag{4.28}$$

$$\begin{cases} u_t = u_{xxx} - 6u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_{xx}U + 6u_xU_x - 12uu_xU - 6u^2U_x + \langle U, U\rangle U_x + 4\langle U, U_x\rangle U, \end{cases} \tag{4.29}$$

$$\begin{cases} u_t = u_{xxx} - 6u^2u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -2U_{xxx} - 6u_{xx}U - 6u_xU_x + 12uu_xU + 6u^2U_x + \langle U, U\rangle U_x - 2\langle U, U_x\rangle U. \end{cases} \tag{4.30}$$

All systems (4.6)–(4.30) admit the reduction $U = \mathbf{0}$. From this viewpoint, (4.6)–(4.9), (4.10)–(4.12), (4.13)–(4.19) and (4.20)–(4.30) are generalizations of the third-order Burgers equation, the trivial equation $u_t = 0$, the pKdV equation and the mKdV equation, respectively. Actually, (4.6)–(4.8) are the third-order symmetries of the second-order systems (4.3)–(4.5), respectively.

4.2. Integrability of systems (4.3)–(4.30)

4.2.1. Systems (4.3) and (4.6). We investigate the second-order system (4.3) and its third-order symmetry (4.6) together. We note that the linear term u_{xx} in (4.3) vanishes iff $a = -\frac{1}{2}$, while the term u_{xxx} in (4.6) vanishes iff $a = 0$. This indicates that the case distinctions of $a \neq -\frac{1}{2}$ or $a = -\frac{1}{2}$ and $a \neq 0$ or $a = 0$ are not very essential for the whole hierarchy of systems starting from (4.3). As an extension of the Hopf–Cole transformation, we consider the change of variables

$$\begin{cases} w = \exp\left(\int^x u\,\mathrm{d}x'\right), \\ W = U\exp\left(\frac{1}{2}\int^x u\,\mathrm{d}x'\right). \end{cases}$$

Then we can triangular linearize (4.3) and (4.6) simultaneously as

$$\begin{cases} w_t = \frac{1}{3}(1+2a)w_{xx} + \frac{2}{3}\langle W, W\rangle, \\ W_t = W_{xx} \end{cases} \tag{4.31}$$

and

$$\begin{cases} w_t = aw_{xxx} + \langle W, W\rangle_x, \\ W_t = W_{xxx}. \end{cases} \tag{4.32}$$

For some values of a, we can solve these systems easily. When $a = 1$, we can fully linearize systems (4.31) and (4.32) through defining new variables V and v by $W = V_x$, $w + \frac{1}{3}\langle V, V\rangle = v$ (see [17]). When $a = -\frac{1}{2}$, we integrate the equation for w in (4.31) to obtain

$$w(x,t) = w(x,0) + \frac{2}{3}\int_0^t \langle W(x,t'), W(x,t')\rangle\,\mathrm{d}t'.$$

Similarly, when $a = 0$, we can integrate the equation for w in (4.32). We mention that Beukers, Sanders and Wang [37, 38] studied higher symmetries of the triangular linear systems (4.31) and (4.32) in the case of scalar W.

Symmetrization. We discuss symmetrization for the second-order system (4.3), since it is more fundamental than its third-order symmetry (4.6). To identify (4.3) as a multi-component generalization of a system in [18, 26], we assume the condition $a \neq -\frac{1}{2}$ and rescale variables as

$$\partial_t = \tfrac{1}{3}(1+2a)\partial_s, \qquad u = 4u', \qquad U = \sqrt{6}U'.$$

In addition, we introduce a new parameter α by the relation

$$\frac{3}{1+2a} = 1 - 2\alpha,$$

where $\alpha \neq \frac{1}{2}$. Then, (4.3) is rewritten as

$$\begin{cases} u'_s = u'_{xx} + 8u'u'_x + (2-4\alpha)\langle U', U'_x\rangle, \\ U'_s = (1-2\alpha)U'_{xx} - 4\alpha u'_x U' + (4-8\alpha)u'U'_x - (4+8\alpha)u'^2U' \\ \quad - (2-4\alpha)\langle U', U'\rangle U'. \end{cases}$$

In the case where U' is scalar, this system is identical to (3.7) in [18]. In that case, considering the linear change of variables

$$u' = q + r, \qquad U' = q - r,$$

we obtain a system of two symmetrically coupled Burgers equations, which coincides with (3.6) in [18] or (3.7) in [26]. We note that system (4.3) with $a = -\frac{1}{2}$ does not have any counterpart in [18, 26], because of its degeneracy of the linear part (cf the introduction).

4.2.2. Systems (4.4) and (4.7). We investigate the second-order system (4.4) and its third-order symmetry (4.7) together. Here, we note that (4.4) and (4.7) are obtained from (4.3) and (4.6), respectively, by rescaling t, U appropriately and taking the limit $a \to \infty$. Then, through the same change of variables as in section 4.2.1,

$$\begin{cases} w = \exp\left(\int^x u \, \mathrm{d}x'\right), \\ W = U \exp\left(\frac{1}{2}\int^x u \, \mathrm{d}x'\right), \end{cases}$$

we can transform systems (4.4) and (4.7) to

$$\begin{cases} w_t = w_{xx} + \langle W, W\rangle, \\ W_t = \mathbf{0} \end{cases} \tag{4.33}$$

and

$$\begin{cases} w_t = w_{xxx} + \langle W, W\rangle_x, \\ W_t = \mathbf{0}. \end{cases} \tag{4.34}$$

Moreover, introducing a function $g(x)$ such that $g''(x) = \langle W, W\rangle$, we can linearize the equations for w in (4.33) and (4.34) with respect to the variable $w + g(x)$.

Symmetrization. We discuss symmetrization for the second-order system (4.4). To identify (4.4) as a multi-component generalization of a system in [18, 26], we rescale the dependent variables as

$$u = 4u', \qquad U = 2\sqrt{5}U'.$$

Then, (4.4) is rewritten as

$$\begin{cases} u'_t = u'_{xx} + 8u'u'_x + 10\langle U', U'_x\rangle, \\ U'_t = -2u'_x U' - 8u'^2 U' - 10\langle U', U'\rangle U'. \end{cases}$$

In the case where U' is scalar, this system should coincide with (3.10) in [18], if it were written correctly. Unfortunately, in [18], Foursov made a mistake in deriving the equation for z in (3.10) from (3.9). It should be corrected as $z_t = -2w_x z - 8w^2 z - 10z^3$. If we consider the linear change of variables

$$u' = q + r, \qquad U' = q - r,$$

we obtain a system of two symmetrically coupled Burgers equations, which coincides with (3.9) in [18] or (3.6) in [26].

4.2.3. Systems (4.5) and (4.8). We concentrate our attention on the second-order system (4.5), and do not study its third-order symmetry (4.8). Defining a new variable w by $w \equiv \frac{1}{2}\langle U, U\rangle$, we find that (4.5) contains a two-component Burgers system

$$\begin{cases} u_t = u_{xx} + 2uu_x + w_x, \\ w_t = (uw)_x. \end{cases} \tag{4.35}$$

Therefore, system (4.5) is a triangular system that consists of the Burgers system (4.35) and the linear equation for U with Burgers-system-dependent coefficients. We mention that in the long-wave limit (disappearance of u_{xx}), (4.35) reduces to the Leroux system and that (4.35) can be rewritten as a non-evolutionary scalar equation in (at least) two different ways. The symmetry integrability of (4.35) as well as the existence of a recursion operator has already been demonstrated [18, 62]. Nevertheless, we could find neither a linearizing transformation nor a *true* Lax representation for (4.35). In what follows, we discuss travelling-wave solutions of (4.35), which are expected to give useful information on its properties.

Substituting the travelling-wave form

$$u(x,t) = f(z) - a, \qquad w(x,t) = g(z), \qquad z = x - at$$

into (4.35), we get a system of two ordinary differential equations. Integrating it once, we obtain

$$\begin{cases} f' + f^2 - af + g + b = 0, \\ fg + c = 0. \end{cases} \tag{4.36}$$

Here, b and c are integration constants that are determined from the boundary conditions for u and w. Plunging $g = -c/f$ into the first equation in (4.36), we obtain the ordinary differential equation for f,

$$\frac{\mathrm{d}f}{\mathrm{d}z} = -\frac{f^3 - af^2 + bf - c}{f}. \tag{4.37}$$

For the sake of simplicity, we assume that $f^3 - af^2 + bf - c$ can be factorized into the product $(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)$ with three distinct real roots $\alpha_1, \alpha_2, \alpha_3$. Thus, we have

$$a = \alpha_1 + \alpha_2 + \alpha_3, \qquad b = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1, \qquad c = \alpha_1\alpha_2\alpha_3.$$

Furthermore, we assume the conditions $\alpha_j \neq 0\,(j = 1, 2, 3)$ to obtain nontrivial solutions of (4.35). Indeed, if $\alpha_j = 0$, then $c = 0$ and we obtain from (4.36) either a trivial solution or a solution of the scalar Burgers equation. Noting the identity

$$\frac{f}{(f-\alpha_1)(f-\alpha_2)(f-\alpha_3)} = \frac{\alpha_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}\left(\frac{1}{f-\alpha_1} - \frac{1}{f-\alpha_3}\right) + \frac{\alpha_2}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}\left(\frac{1}{f-\alpha_2} - \frac{1}{f-\alpha_3}\right),$$

we can integrate (4.37) to obtain

$$\left(1 + \frac{\alpha_3-\alpha_1}{f-\alpha_3}\right)^{\frac{\alpha_1}{(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}} \left(1 + \frac{\alpha_3-\alpha_2}{f-\alpha_3}\right)^{\frac{\alpha_2}{(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}} = d\mathrm{e}^{-z}, \tag{4.38}$$

where d is a constant. Now, it is clear that the functional form of $1/(f-\alpha_3)$ depends on the ratio of two powers on the left-hand side, $\left(\alpha_3^{-1} - \alpha_2^{-1}\right)/\left(\alpha_1^{-1} - \alpha_3^{-1}\right)$. Let us consider the simplest case in which this ratio is unity, i.e.

$$\frac{1}{\alpha_3} = \frac{1}{2}\left(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\right).$$

In this case, we have $\alpha_1 + \alpha_2 \neq 0$ for the existence of α_3. Then we can solve (4.38) explicitly for $1/(f-\alpha_3)$:

$$\frac{1}{f-\alpha_3} = -\frac{\alpha_1 + \alpha_2 + \frac{(\alpha_1+\alpha_2)^2}{\alpha_1-\alpha_2}\sqrt{1 + \exp\left[-\frac{(\alpha_1-\alpha_2)^2}{\alpha_1+\alpha_2}(z-z_0)\right]}}{2\alpha_1\alpha_2}. \tag{4.39}$$

Here, z_0 is the constant given by

$$\mathrm{e}^{z_0} \equiv d\left[\frac{-4\alpha_1\alpha_2}{(\alpha_1+\alpha_2)^2}\right]^{\frac{\alpha_1+\alpha_2}{(\alpha_1-\alpha_2)^2}}.$$

Using the relation $\alpha_3 = 2\alpha_1\alpha_2/(\alpha_1+\alpha_2)$, we can rewrite (4.39) as

$$f(z) = \frac{2\alpha_1\alpha_2}{\alpha_1+\alpha_2+\frac{\alpha_1-\alpha_2}{\sqrt{1+\exp\left[-\frac{(\alpha_1-\alpha_2)^2}{\alpha_1+\alpha_2}(z-z_0)\right]}}}. \tag{4.40}$$

In order for the function $f(z)$ to be non-singular, we should assume the condition $\alpha_1(\alpha_1+\alpha_2) > 0$. To summarize, we have obtained, in the simplest case, a travelling-wave solution of (4.35) given by

$$u(x,t) = f(x-at) - a, \qquad w(x,t) = -\frac{c}{f(x-at)},$$

with (4.40), $a = \alpha_1+\alpha_2+2\alpha_1\alpha_2/(\alpha_1+\alpha_2)$ and $c = 2(\alpha_1\alpha_2)^2/(\alpha_1+\alpha_2)$.

When a nontrivial solution, like the above, of subsystem (4.35) is given, we can solve the remaining equation for U in the original system (4.5), $\left(U_j^2\right)_t = \left(uU_j^2\right)_x$, in the same way as in section 3.2.3.

Symmetrization. We discuss symmetrization for the second-order system (4.5). With the rescaling of dependent variables

$$u = 2u', \qquad U = \sqrt{6}U',$$

(4.5) is rewritten as

$$\begin{cases} u'_t = u'_{xx} + 4u'u'_x + 3\langle U', U'_x\rangle, \\ U'_t = u'_x U' + 2u'U'_x. \end{cases}$$

In the case where U' is scalar, this system is identical to (3.4) in [18]. If we consider the linear change of variables

$$u' = q + r, \qquad U' = q - r,$$

we obtain a system of two symmetrically coupled Burgers equations, which coincides with (3.3) in [18] or (3.5) in [26].

4.2.4. System (4.9). In the case where U is scalar, system (4.9) coincides with system (4.8). However, unlike (4.8), (4.9) in the general N case does not possess a second-order symmetry of the form (4.1). System (4.9) contains the third-order symmetry of the two-component Burgers system (4.35), where w is again given by $w = \frac{1}{2}\langle U, U\rangle$. Therefore, in order to demonstrate the integrability of (4.9), we first of all need to know either a linearizing transformation or a Lax representation for (4.35). This remains an open question.

Symmetrization. Through symmetrization of (4.9) in the case of scalar U, we just obtain the third-order symmetry of the two symmetrically coupled Burgers equations in section 4.2.3.

4.2.5. System (4.10). System (4.10) is connected with system (4.12) through a Miura-type transformation. We discuss the integrability of these two systems in section 4.2.7.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{\alpha}(q - r).$$

Here, α is a nonzero constant. Then we can rewrite (4.10) as a system of two symmetrically coupled equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} - \frac{1}{2}r_{xxx} + \frac{1}{2}(1+3\alpha)(q-r)q_{xx} + \frac{1}{2}(1-3\alpha)(q-r)r_{xx} \\ \qquad + \frac{1}{2}q_x^2 - \frac{1}{2}r_x^2 + 3\alpha(q-r)^2 r_x - \frac{3}{2}\alpha^2(q-r)^4, \\ r_t = -\frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} - \frac{1}{2}(1-3\alpha)(q-r)q_{xx} - \frac{1}{2}(1+3\alpha)(q-r)r_{xx} \\ \qquad - \frac{1}{2}q_x^2 + \frac{1}{2}r_x^2 + 3\alpha(q-r)^2 q_x - \frac{3}{2}\alpha^2(q-r)^4. \end{cases}$$

This system does not belong to the class of systems studied in [25, 26], because of its degeneracy of the linear part.

4.2.6. System (4.11). For system (4.11), we have the relation $(u_x - \langle U, U\rangle)_t = 0$. Thus, we can set

$$u_x - \langle U, U\rangle \equiv \phi(x), \tag{4.41}$$

where the function $\phi(x)$ does not depend on t. Then, the equation for U is rewritten in terms of $\phi(x)$ as

$$U_t = U_{xxx} + 2\phi U_x + \phi_x U. \tag{4.42}$$

The solutions of (4.42) are given by

$$U(x,t) = \int \mathrm{d}\lambda\, \mathrm{e}^{\lambda t}\Psi(x;\lambda),$$

where $\Psi(x;\lambda)$ is a solution of the ordinary differential equation

$$\Psi_{xxx} + 2\phi\Psi_x + \phi_x\Psi = \lambda\Psi. \tag{4.43}$$

Once we obtain $\phi(x)$ and $U(x,t)$, we can determine $u(x,t)$ by using (4.41). The vector equation (4.43) is of the same form as the scattering problem associated with the Kaup–Kupershmidt equation [63, 64]. We can see that this is not a coincidence through investigation of the fifth-order symmetry of system (4.11). Indeed, the fifth-order symmetry is rewritten (up to a scaling of t_5) in terms of ϕ and Ψ as

$$\begin{cases} \phi_{t_5} + \phi_{xxxxx} + 10\phi\phi_{xxx} + 25\phi_x\phi_{xx} + 20\phi^2\phi_x = 0, \\ \Psi_{t_5} = 9\Psi_{xxxxx} + 30\phi\Psi_{xxx} + 45\phi_x\Psi_{xx} + (35\phi_{xx} + 20\phi^2)\Psi_x + (10\phi_{xxx} + 20\phi\phi_x)\Psi. \end{cases}$$

The first equation is nothing but the Kaup–Kupershmidt equation, while the second equation together with (4.43) constitutes a Lax representation for it.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{\alpha}(q - r).$$

Here, α is a nonzero constant. Then we can rewrite (4.11) as a system of two symmetrically coupled equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} - \frac{1}{2}r_{xxx} + \left(\frac{1}{2} + \alpha\right)(q-r)q_{xx} + \left(\frac{1}{2} - \alpha\right)(q-r)r_{xx} \\ \qquad + \left(1 - \frac{1}{2}\alpha\right)q_x^2 + \alpha q_x r_x - \left(1 + \frac{1}{2}\alpha\right)r_x^2 - \alpha(q-r)^2 q_x \\ \qquad + 3\alpha(q-r)^2 r_x - \alpha^2(q-r)^4, \\ r_t = -\frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} - \left(\frac{1}{2} - \alpha\right)(q-r)q_{xx} - \left(\frac{1}{2} + \alpha\right)(q-r)r_{xx} \\ \qquad - \left(1 + \frac{1}{2}\alpha\right)q_x^2 + \alpha q_x r_x + \left(1 - \frac{1}{2}\alpha\right)r_x^2 + 3\alpha(q-r)^2 q_x \\ \qquad - \alpha(q-r)^2 r_x - \alpha^2(q-r)^4. \end{cases}$$

This system does not belong to the class of nondegenerate systems studied in [25, 26].

4.2.7. System (4.12). For system (4.12), if we define new variables w and W by

$$\begin{cases} w \equiv -u_x - \frac{1}{2}u^2 + \frac{1}{2}\langle U, U\rangle, \\ W \equiv U_x + uU, \end{cases} \tag{4.44}$$

they satisfy the following system:

$$\begin{cases} w_t = -3\langle W, W_x\rangle, \\ W_t = W_{xxx} + w_x W + 2wW_x. \end{cases} \tag{4.45}$$

This system coincides with the multi-component Drinfel'd–Sokolov system (3.3), up to a scaling of W. The Miura map (4.44) is a multi-component generalization of that for the case of scalar U in [43, 44].

Relation to system (4.10). If we introduce a new scalar variable v by

$$v \equiv u_x - u^2 + \langle U, U\rangle, \tag{4.46}$$

system (4.12) is changed into the following system:

$$\begin{cases} v_t = (3v\langle U, U\rangle + 3\langle U, U_{xx}\rangle - 3\langle U, U\rangle^2)_x, \\ U_t = U_{xxx} + v_x U + vU_x - 3\langle U, U_x\rangle U. \end{cases} \tag{4.47}$$

Then, it is straightforward to obtain (4.10) (for $\hat{u}$ and U) from (4.47) through potentiation $v = \hat{u}_x$. Combining (4.46) and (4.44), we obtain the relation $v - 2w = 3u_x$, and consequently,

$$\hat{u} - 2\int^x w \,\mathrm{d}x' = 3u.$$

Using this relation, we can also rewrite (4.44) as a transformation between system (4.10) and the multi-component Drinfel'd–Sokolov system (4.45).

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{\alpha}(q - r).$$

Here, α is a nonzero constant. Then we can rewrite (4.12) as a system of two symmetrically coupled equations

$$\begin{cases} q_t = \frac{1}{2}[q_{xx} - r_{xx} + (1+\alpha)(q-r)q_x + (1-\alpha)(q-r)r_x \\ \qquad - (1-\alpha)q^3 - (1+\alpha)q^2 r + (1-\alpha)qr^2 + (1+\alpha)r^3]_x, \\ r_t = \frac{1}{2}[-q_{xx} + r_{xx} - (1-\alpha)(q-r)q_x - (1+\alpha)(q-r)r_x \\ \qquad + (1+\alpha)q^3 + (1-\alpha)q^2 r - (1+\alpha)qr^2 - (1-\alpha)r^3]_x. \end{cases} \tag{4.48}$$

This system does not belong to the class of nondegenerate systems studied in [25, 26]. However, it was found in connection with the Kac–Moody Lie algebras and written in a Hamiltonian form about 20 years ago [43, 44]. More specifically, system (4.48) with $\alpha = -1$ coincides with (3)–(4) in [43] for the $D_3^{(2)}$ case, up to a scaling of variables (see also the generalized mKdV equation in [44] for the $A_3^{(2)}$ case).

4.2.8. System (4.13). System (4.13) is merely a potential form of the multi-component Zakharov–Ito system (3.5).

Symmetrization. In the case where U is scalar, we set

$$u = q + r, \qquad U = q - r.$$

Then we can rewrite (4.13) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + 2q_x^2 + r_x^2, \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + q_x^2 + 2r_x^2. \end{cases}$$

This system is identical to (37) in [25] or (3.10) in [26].

4.2.9. System (4.14). *Remark.* If we consider separately the two cases $a = 0$ and $a \neq 0$, we can reduce the number of parameters in system (4.14) by scaling variables. In the former case the parameter b can also be fixed at any nonzero value, while in the latter case only the parameter a can be scaled away. However, this case distinction is neither necessary nor essential, as is demonstrated below.

If we define new variables w and W by

$$\begin{cases} w \equiv u_x + \frac{a}{2}\langle U, U\rangle, \\ W \equiv \sqrt{\langle U, U\rangle}U, \end{cases}$$

they satisfy the following system:

$$\begin{cases} w_t = w_{xxx} + 6ww_x + \left(b - \frac{a^2}{4}\right)\langle W, W\rangle_x, \\ W_t = 2(wW)_x. \end{cases} \tag{4.49}$$

Thus, the essential parameter is $b - a^2/4$ rather than a or b. If $b - a^2/4 \neq 0$, system (4.49) coincides with the multi-component Zakharov–Ito system (3.5), up to a scaling of variables. When $b - a^2/4 = 0$, (4.49) is a triangular system that consists of the KdV equation and the linear equation for W with KdV-equation-dependent coefficients. This triangular system was studied from a point of view of symmetries in [65] (see also [22, 66, 67]). As we have noted in section 3.2.3, we can relate to W the inverse square of a solution of the KdV linear problem.

Symmetrization. In the case where U is scalar, we set

$$u = q + r, \qquad U = q - r, \qquad a = 1 + 2\alpha, \qquad b = 2\beta.$$

Then we can rewrite (4.14) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + (1+\alpha)(q-r)q_{xx} - \alpha(q-r)r_{xx} + (3+\alpha)q_x^2 \\ \qquad + (2-2\alpha)q_xr_x + (1+\alpha)r_x^2 + (2+4\alpha)(q-r)^2q_x + \beta(q-r)^4, \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \alpha(q-r)q_{xx} - (1+\alpha)(q-r)r_{xx} + (1+\alpha)q_x^2 \\ \qquad + (2-2\alpha)q_xr_x + (3+\alpha)r_x^2 + (2+4\alpha)(q-r)^2r_x + \beta(q-r)^4. \end{cases} \tag{4.50}$$

This system is identical to (12) in [25]. Foursov claimed in [25] that this system was either reduced to the representative case $\alpha = 0$ ($a = 1$) or decoupled by a linear change of dependent variables. However, in fact this is not true. As far as we consider a linear change of variables, we need one more representative case, $\alpha = -\frac{1}{2}$ ($a = 0$), in which (4.50) cannot be decoupled. Therefore, we can say that (at least) one system is missing from the final list of Foursov–Olver given in [26].

4.2.10. System (4.15). System (4.15) is merely a potential form of the Jordan KdV system (3.4). We see in section 4.2.22 that this system is also connected with system (4.27) through a Miura-type transformation.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = \frac{1}{2}(q + r), \qquad U = \frac{\mathrm{i}}{2}(q - r).$$

Then, (4.15) is decoupled into two pKdV equations

$$\begin{cases} q_t = q_{xxx} + 3q_x^2, \\ r_t = r_{xxx} + 3r_x^2. \end{cases}$$

This corresponds to (27) in [25].

4.2.11. System (4.16). System (4.16) is connected with system (4.28) through a Miura-type transformation. We discuss the integrability of these two systems in section 4.2.23.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{3}(q - r).$$

Then we can rewrite (4.16) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = q_{xxx} + 3(q - r)q_{xx} + 3(q_x + r_x)q_x + 3(q - r)^2 q_x, \\ r_t = r_{xxx} - 3(q - r)r_{xx} + 3(q_x + r_x)r_x + 3(q - r)^2 r_x. \end{cases}$$

This system coincides with (34) in [25] or (3.9) in [26].

4.2.12. System (4.17). System (4.17) is connected with system (4.29) through a Miura-type transformation. We discuss the integrability of these two systems in section 4.2.24.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = \tfrac{1}{2}(q + r), \qquad U = \tfrac{\sqrt{6}}{2}(q - r).$$

Then we can rewrite (4.17) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = q_{xxx} + 3(q - r)q_{xx} + 3q_x^2 + 3(q - r)^2 q_x, \\ r_t = r_{xxx} - 3(q - r)r_{xx} + 3r_x^2 + 3(q - r)^2 r_x. \end{cases}$$

This system coincides with (28) in [25] or (3.8) in [26].

4.2.13. System (4.18). System (4.18) is connected with system (4.30) through a Miura-type transformation. We discuss the integrability of these two systems in section 4.2.25.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{3}(q - r).$$

Then we can rewrite (4.18) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = -\tfrac{1}{2}q_{xxx} + \tfrac{3}{2}r_{xxx} - 6(q - r)r_{xx} + 6r_x^2 + 12(q - r)^2 r_x + 3(q - r)^4, \\ r_t = \tfrac{3}{2}q_{xxx} - \tfrac{1}{2}r_{xxx} + 6(q - r)q_{xx} + 6q_x^2 + 12(q - r)^2 q_x + 3(q - r)^4. \end{cases}$$

This system is identical to (13) in [25] with $\alpha = 6$. Thus, it is equivalent to (41) in [25] or (3.12) in [26], up to a linear change of dependent variables.

4.2.14. System (4.19). For system (4.19), if we introduce a new variable w by

$$w \equiv u_x + 2\langle U, U\rangle, \tag{4.51}$$

it solves the KdV equation

$$w_t = w_{xxx} + 2ww_x. \tag{4.52a}$$

Therefore, system (4.19) is reduced to a triangular form. The equation for U is rewritten in terms of w as

$$U_t = 4U_{xxx} + w_x U + 2wU_x. \tag{4.52b}$$

We note that the vector equation (4.52*b*) is of the same form as the time part of the linear problem for the KdV equation (4.52*a*). This relation between system (4.19) and the KdV equation resembles the relation between the fifth-order symmetry of system (4.11) and the Kaup–Kupershmidt equation shown in section 4.2.6. A recursion operator and a Lax representation for the triangular system (4.52) were given in [67] and [24], respectively. Once we obtain $w(x,t)$ and $U(x,t)$, we can determine $u(x,t)$ by using (4.51).

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = 6(q+r), \qquad U = \sqrt{3}\mathrm{i}(q-r).$$

Then we can rewrite (4.19) as a system of two symmetrically coupled pKdV equations

$$\begin{cases} q_t = \frac{5}{2}q_{xxx} - \frac{3}{2}r_{xxx} + 6(q-r)q_{xx} + 6q_x^2 + 12q_x r_x - 6r_x^2 \\ \qquad - 12(q-r)^2(q_x - r_x) - 3(q-r)^4, \\ r_t = -\frac{3}{2}q_{xxx} + \frac{5}{2}r_{xxx} - 6(q-r)r_{xx} - 6q_x^2 + 12q_x r_x + 6r_x^2 \\ \qquad + 12(q-r)^2(q_x - r_x) - 3(q-r)^4. \end{cases}$$

This system coincides with (42) in [25] or (3.13) in [26].

4.2.15. System (4.20). For system (4.20), if we introduce a new variable w by

$$w \equiv -u_x + \tfrac{1}{2}u^2 - \tfrac{1}{2}\langle U, U\rangle, \tag{4.53}$$

it solves the KdV equation

$$w_t = w_{xxx} - 3ww_x. \tag{4.54a}$$

Therefore, system (4.20) is reduced to a triangular form. Still, it is a very interesting system. To see this, we rewrite the equation for u in terms of w as

$$u_t = -2u_x^2 + u^2 u_x - 3wu_x - w_x u - w_{xx}. \tag{4.54b}$$

Then, the reduction $w = 0$ changes this equation to a nontrivial closed equation for u. With a rescaling of variables, it reads

$$u_t = u_x(u_x - u^2). \tag{4.55}$$

Equation (4.55) possesses an infinite set of commuting symmetries

$$u_{\tau_n} = u_x(u_x - u^2)^n, \qquad n \in \mathbb{R}.$$

We can easily obtain a travelling-wave solution of (4.55) with two arbitrary constants [68], which is called a complete solution in the theory of partial differential equations. However, we do not know any explicit formula for the general solution of (4.55). Using (4.53), we can rewrite the equation for U as a linear equation with a (u, w)-dependent coefficient.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = q - r.$$

Then we can rewrite (4.20) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{2}(q-r)(q_{xx} - r_{xx}) - (q_x - r_x)r_x \\ \qquad + (q^2 - 5qr)q_x - (q^2 + qr)r_x, \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{2}(q-r)(q_{xx} - r_{xx}) + (q_x - r_x)q_x \\ \qquad - (qr + r^2)q_x + (-5qr + r^2)r_x. \end{cases}$$

This system coincides with (57) in [25] or (3.18) in [26], up to a scaling of variables.

4.2.16. System (4.21). In system (4.21), u and $\langle U, U\rangle$ satisfy the same equations as in system (4.20). Thus, if we define w by (4.53), we obtain (4.54*a*) and (4.54*b*) again. The only difference between the two systems (4.21) and (4.20) lies in the forms of equations for U, which can be rewritten as linear equations with (u, w)-dependent coefficients.

Symmetrization. In the case where U is scalar, (4.21) is identical to (4.20). Thus, through symmetrization, we just obtain the same result as in section 4.2.15.

4.2.17. System (4.22). System (4.22) has already been obtained in [69] as a reduction of a bi-Hamiltonian system (see also [70]). If we introduce a new variable w by

$$w \equiv -u_x + \tfrac{1}{2}u^2 - \tfrac{1}{2}\langle U, U\rangle,$$

it solves the KdV equation

$$w_t = w_{xxx} - 3ww_x. \tag{4.56}$$

Therefore, system (4.22) is reduced to a triangular form. Substituting $\frac{1}{2}\langle U, U\rangle = -u_x + \frac{1}{2}u^2 - w$ into the equations for u and U respectively, we obtain

$$\begin{cases} u_t = -(wu + w_x)_x, \\ U_t = -(wU)_x. \end{cases}$$

Thus, the reduction $w = 0$ is trivial in this system. We mention again (cf section 4.2.9) that the above equation for U coupled to the KdV equation (4.56) was studied in [65].

Remark. Actually, a one-parameter deformation of system (4.22),

$$\begin{cases} u_t = 3au_x + u_{xxx} - \frac{3}{2}u^2u_x + \frac{1}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = aU_x + u_{xx}U + u_xU_x - uu_xU - \frac{1}{2}u^2U_x + \frac{1}{2}\langle U, U\rangle U_x + \langle U, U_x\rangle U, \end{cases} \tag{4.57}$$

is still integrable. Indeed, if we introduce w in this case by

$$w \equiv -a - u_x + \tfrac{1}{2}u^2 - \tfrac{1}{2}\langle U, U\rangle,$$

system (4.57) is changed into the following system:

$$\begin{cases} w_t = w_{xxx} - 3ww_x + 2a\langle U, U_x\rangle, \\ U_t = -(wU)_x. \end{cases}$$

When $a \neq 0$, this system coincides with the multi-component Zakharov–Ito system (3.5), up to a scaling of variables.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = q - r.$$

Then we can rewrite (4.22) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + (q - r)q_{xx} + (q_x - r_x)q_x - 4qrq_x - 2q^2r_x, \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} - (q - r)r_{xx} - (q_x - r_x)r_x - 2r^2q_x - 4qrr_x. \end{cases}$$

This system coincides with (58) in [25] or (3.19) in [26].

4.2.18. System (4.23). For system (4.23), if we introduce a new variable w by

$$w \equiv -u_x + \tfrac{1}{2}u^2 - \tfrac{1}{2}\langle U, U\rangle,$$

it satisfies the KdV equation

$$w_t = w_{xxx} - 3ww_x. \tag{4.58a}$$

Therefore, system (4.23) is reduced to a triangular form. The equation for u is rewritten in terms of w as

$$u_t = -u^2u_x + \tfrac{1}{2}u^4 - w_{xx} + wu_x - w_xu - 2wu^2 + 2w^2. \tag{4.58b}$$

The triangular system (4.58) possesses (at least) two higher symmetries. The first higher symmetry is given by

$$\begin{cases} w_{t_5} = w_{xxxxx} - 5ww_{xxx} - 10w_x w_{xx} + \frac{15}{2}w^2 w_x, \\ u_{t_5} = -\frac{1}{2}u^4 w + 2u^2 w^2 - 2w^3 + u^2 w u_x - \frac{1}{2}w^2 u_x - u^3 w_x + 5uww_x \\ \qquad + 2uu_x w_x + 3w_x^2 - u^2 w_{xx} + 5ww_{xx} + u_x w_{xx} - uw_{xxx} - w_{xxxx}, \end{cases}$$

which obviously vanishes under the reduction $w = 0$. Similarly, the second one vanishes under the same reduction. On the other hand, the reduction $w = 0$ changes (4.58) to a nontrivial closed equation for u,

$$u_t = -u^2 u_x + \tfrac{1}{2}u^4.$$

As far as we could check with the help of a computer, this equation seems to have no polynomial higher symmetry. We can construct its general solution in implicit form using the method of characteristic curves.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = q - r.$$

Then we can rewrite (4.23) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{2}(q-r)(q_{xx} - r_{xx}) - (q_x - r_x)r_x \\ \qquad - (3qr + r^2)q_x + (-3qr + r^2)r_x + \frac{1}{2}(q-r)^3 q, \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{2}(q-r)(q_{xx} - r_{xx}) + (q_x - r_x)q_x \\ \qquad + (q^2 - 3qr)q_x - (q^2 + 3qr)r_x - \frac{1}{2}(q-r)^3 r. \end{cases}$$

This system coincides with (61) in [25] or (3.20) in [26], up to a scaling of variables.

4.2.19. System (4.24). For system (4.24), if we introduce a new variable w by

$$w \equiv -u_x + \tfrac{1}{2}u^2 - \tfrac{1}{2}\langle U, U\rangle, \tag{4.59}$$

it satisfies the KdV equation

$$w_t = w_{xxx} - 3ww_x.$$

Therefore, system (4.24) is rewritten in a triangular form. Substituting $\frac{1}{2}\langle U, U\rangle = -u_x + \frac{1}{2}u^2 - w$ into the equations for u and U respectively, we obtain

$$\begin{cases} u_t = -w_x u - \frac{1}{2}wu^2 - w_{xx} + w^2, \\ U_t = -\frac{1}{2}(w_x + wu)U. \end{cases} \tag{4.60}$$

Thus, the reduction $w = 0$ is trivial in this system. We note that in (4.60) no term involves x-derivatives of u, U such as u_x, U_x, u_{xx}, U_{xx}. Then, for a given solution of the KdV equation $w(x,t)$, the equation for $u(x,t)$ can be regarded as a Riccati equation with x fixed. Once we obtain $w(x,t)$ and $u(x,t)$, we can integrate the equation for U as

$$U(x,t) = \exp\left(-\frac{1}{2}\int_0^t (w_x + wu)\,\mathrm{d}t'\right)U(x,0).$$

Remark. Actually, system (4.24) is the third-order symmetry of a nontrivial first-order system,

$$\begin{cases} u_{t_1} = u_x + \frac{1}{2}\langle U, U\rangle, \\ U_{t_1} = \frac{1}{2}uU. \end{cases} \tag{4.61}$$

For this system, w defined by (4.59) obeys the linear equation $w_{t_1} = w_x$ and the equation for u is rewritten as $u_{t_1} = \frac{1}{2}u^2 - w$. System (4.61) in the $N = 1$ case as well as its higher symmetries was studied in [56, 71, 72] (see also [73] for its soliton-like solutions).

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = q - r.$$

Then we can rewrite (4.24) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{4}(q-r)(3q_{xx} - r_{xx}) + \frac{1}{2}(q_x - r_x)^2 \\ \qquad + \frac{1}{2}(q^2 - 6qr - r^2)q_x - (q^2 + 2qr)r_x - q^2r(q-r), \\ r_t = \frac{1}{2}q_{xxx} + \frac{1}{2}r_{xxx} + \frac{1}{4}(q-r)(q_{xx} - 3r_{xx}) + \frac{1}{2}(q_x - r_x)^2 \\ \qquad - (2qr + r^2)q_x - \frac{1}{2}(q^2 + 6qr - r^2)r_x + qr^2(q-r). \end{cases}$$

This system coincides with (23) in [25] with $\alpha = 1$, up to a scaling of dependent variables. Thus, it is equivalent to (62) in [25] or (3.21) in [26], up to a linear change of variables.

4.2.20. System (4.25). System (4.25) is just a disguised form of a single vector equation. Indeed, if we introduce an $(N+1)$-component vector W by $W \equiv (u, U) = (u, U_1, \ldots, U_N)$, system (4.25) can be rewritten in the form

$$W_t = W_{xxx} + \langle W, W\rangle W_x.$$

This is a well-known vector mKdV equation and its integrability has been established in the literature [27, 28, 74, 75].

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = \frac{1}{2}(q + r), \qquad U = \frac{\mathrm{i}}{2}(q - r).$$

Then we can rewrite (4.25) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = q_{xxx} + qrq_x, \\ r_t = r_{xxx} + qrr_x. \end{cases}$$

This system is known as (the non-reduced form of) the complex mKdV equation [76]. It is identical to (43) in [25] or (3.14) in [26] with a correction of misprints.

4.2.21. System (4.26). System (4.26) is also a disguised form of a single vector equation. Indeed, if we introduce an $(N+1)$-component vector W by $W \equiv (u, U) = (u, U_1, \ldots, U_N)$, system (4.26) can be rewritten in the form

$$W_t = W_{xxx} + \langle W, W\rangle W_x + \langle W, W_x\rangle W.$$

This is another well-known vector mKdV equation, for which a Lax representation was given in [77] for the $N = 1$ case and in [78] for the general N case.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \mathrm{i}(q - r).$$

Then we can rewrite (4.26) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = q_{xxx} + 6qrq_x + 2q^2r_x, \\ r_t = r_{xxx} + 2r^2q_x + 6qrr_x. \end{cases} \tag{4.62}$$

This system coincides with (48) in [25] or (3.15) in [26], up to a scaling of variables.

4.2.22. System (4.27). System (4.27) is known as a Jordan mKdV system [27]. Let us briefly summarize its integrability. It is well known that the matrix mKdV equation,

$$Q_t = Q_{xxx} - 3(Q_x Q^2 + Q^2 Q_x), \tag{4.63}$$

admits a Lax representation [52, 75, 78, 79]. Then, system (4.27) is also integrable, because it is obtained from (4.63) through the following reduction:

$$Q = u\mathbf{1} + \sum_{j=1}^{N} U_j \mathbf{e}_j, \qquad \{\mathbf{e}_i, \mathbf{e}_j\}_+ = -2\delta_{ij}\mathbf{1}.$$

We mention that (4.27) admits a generalization to a system for two vector unknowns preserving the integrability [29].

Relation to systems (3.4) and (4.15). If we define new variables w and W by [27]

$$\begin{cases} w \equiv \pm u_x - u^2 + \langle U, U\rangle, \\ W \equiv U_x \mp 2uU, \end{cases}$$

they satisfy the following system:

$$\begin{cases} w_t = w_{xxx} + 3(w^2 - \langle W, W\rangle)_x, \\ W_t = W_{xxx} + 6(wW)_x. \end{cases}$$

This system coincides with the Jordan KdV system (3.4) and, through potentiation of it, we obtain system (4.15).

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = \frac{1}{2}(q + r), \qquad U = \frac{\mathrm{i}}{2}(q - r).$$

Then (4.27) is decoupled into two mKdV equations

$$\begin{cases} q_t = q_{xxx} - 6q^2 q_x, \\ r_t = r_{xxx} - 6r^2 r_x. \end{cases}$$

This corresponds to (50) in [25].

4.2.23. System (4.28). We note that through introduction of a new scalar variable w by

$$w \equiv u_x - u^2,$$

system (4.28) is transformed to a system of coupled KdV–mKdV type,

$$\begin{cases} w_t = w_{xxx} + 6ww_x + w_x\langle U, U\rangle + 2w\langle U, U\rangle_x + \frac{1}{2}\langle U, U\rangle_{xxx}, \\ U_t = U_{xxx} + 3(wU)_x + \langle U, U\rangle U_x + \frac{3}{2}\langle U, U\rangle_x U. \end{cases} \tag{4.64}$$

Let us demonstrate that system (4.28) admits a Lax representation. We consider a pair of linear equations for a column-vector function ψ,

$$\psi_x = \hat{U}\psi, \qquad \psi_t = \hat{V}\psi,$$

with the matrices $\hat{U}$ and $\hat{V}$ of the following form:

$$\hat{U} = \begin{pmatrix} -\mathrm{i}\zeta I_l & Q \\ R & \mathrm{i}\zeta I_m + P \end{pmatrix}, \tag{4.65a}$$

$$\hat{V} = \left(\begin{array}{l|l} -4\mathrm{i}\zeta^3 I_l - 2\mathrm{i}\zeta QR & 4\zeta^2 Q + 2\mathrm{i}\zeta(Q_x + QP) - Q_{xx} \\ + Q_x R - QR_x + 2QPR & -2Q_x P - QP_x + 2QRQ - QP^2 \\ \hline 4\zeta^2 R + 2\mathrm{i}\zeta(-R_x + PR) & 4\mathrm{i}\zeta^3 I_m + 2\mathrm{i}\zeta RQ - P_{xx} + R_x Q \\ - R_{xx} + P_x R + 2PR_x & - RQ_x + P_x P - PP_x + 2PRQ \\ + 2RQR - P^2 R & + 2RQP - P^3 - 3g_x P \end{array}\right). \tag{4.65b}$$

Here, ζ is the spectral parameter, I_l and I_m are the $l \times l$ and $m \times m$ unit matrices, respectively, Q, R and P are $l \times m$, $m \times l$ and $m \times m$ matrices, respectively, and g is a scalar function. The compatibility condition $\psi_{xt} = \psi_{tx}$ implies the so-called zero-curvature condition,

$$\hat{U}_t - \hat{V}_x + \hat{U}\hat{V} - \hat{V}\hat{U} = O.$$

Then, substituting (4.65) into this condition, we obtain a system of three matrix equations

$$\begin{cases} Q_t + Q_{xxx} + 3(Q_x P)_x - 3Q_x R Q - 3QRQ_x + 3Q_x P^2 + 3QP_x P - 3g_x QP \\ \quad = O, \\ R_t + R_{xxx} - 3(PR_x)_x - 3R_x QR - 3RQR_x + 3P^2 R_x + 3PP_x R + 3g_x PR \\ \quad = O, \\ P_t + P_{xxx} + 3(g_x P)_x - 3(PRQ + RQP)_x + 3PP_x P + 3P^2 RQ - 3RQP^2 \\ \quad = O. \end{cases} \tag{4.66}$$

We note that this system admits the reduction $R = {}^tQ, {}^tP = -P$, where the superscript ^{t} denotes the transposition. In particular, if we choose

$$\begin{cases} Q = (u, W_1, \ldots, W_N) \equiv (u, W), \\ R = \begin{pmatrix} u \\ W_1 \\ \vdots \\ W_N \end{pmatrix} = \begin{pmatrix} u \\ {}^tW \end{pmatrix}, \\ P = \begin{pmatrix} 0 & V_1 & \cdots & V_N \\ -V_1 & & & \\ \vdots & & \boldsymbol{O} & \\ -V_N & & & \end{pmatrix} \equiv \begin{pmatrix} 0 & V \\ -{}^tV & O \end{pmatrix}, \end{cases}$$

system (4.66) is reduced to the system

$$\begin{cases} u_t + u_{xxx} - 6u^2 u_x - 3u_x(\langle W, W\rangle + \langle V, V\rangle) - 3u(\langle W, W_x\rangle + \langle V, V_x\rangle) \\ \quad + 3g_x\langle W, V\rangle - 3\langle W_x, V\rangle_x = 0, \\ W_t + W_{xxx} + 3(u_x V)_x - 3ug_x V - 3uu_x W - 3u^2 W_x - 3\langle W, W\rangle W_x \\ \quad - 3\langle W, W_x\rangle W - 3\langle W, V\rangle_x V = \mathbf{0}, \\ V_t + V_{xxx} + 3(g_x V)_x - 3(u^2 V)_x - 3(\langle W, V\rangle W)_x - 3\langle V, V_x\rangle V \\ \quad - 3u\langle V, V\rangle W + 3u\langle W, V\rangle V = \mathbf{0}, \end{cases} \tag{4.67}$$

together with one constraint

$$(u\,{}^tVW - u\,{}^tWV)_x - \langle W, V\rangle({}^tVW - {}^tWV) = O.$$

When W and V are scalar, i.e. $N = 1$, this constraint is satisfied automatically and we obtain a three-component mKdV system. There is another case, the case $W = V$, for which the constraint is satisfied. Then, if we set

$$g = u, \qquad W = V = \frac{\mathrm{i}}{\sqrt{3}}U,$$

and change the sign of time t ($t \to -t$), system (4.67) collapses to system (4.28).

Relation to system (4.16). If we introduce a new scalar variable v by

$$v \equiv u_x - u^2 + \tfrac{1}{3}\langle U, U\rangle,$$

system (4.28) is changed into the following system:

$$\begin{cases} v_t = (v_{xx} + 3v^2 + v\langle U, U\rangle + \langle U, U_{xx}\rangle)_x, \\ U_t = U_{xxx} + 3v_x U + 3vU_x + \langle U, U_x\rangle U. \end{cases} \tag{4.68}$$

Then, it is straightforward to obtain (4.16) (for $\hat{u}$ and U) from (4.68) through potentiation $v = \hat{u}_x$. It should be mentioned here that the authors encountered two papers [80, 81] on the integrability of (4.68) in the case of scalar U, after they had obtained all the presented results independently.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = q + r, \qquad U = \sqrt{3}(q - r).$$

Then we can rewrite (4.28) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = [q_{xx} + 3(q - r)q_x + q^3 - 12q^2 r + 3qr^2]_x, \\ r_t = [r_{xx} - 3(q - r)r_x + 3q^2 r - 12qr^2 + r^3]_x. \end{cases}$$

This system coincides with (55) in [25] or (3.17) in [26]. Moreover, if we introduce new variables $\hat{q}$ and $\hat{r}$ by

$$\hat{q} \equiv \sqrt{3}\mathrm{i}q \exp\left(\int^x (q - r)\,\mathrm{d}x'\right), \qquad \hat{r} \equiv \sqrt{3}\mathrm{i}r \exp\left(-\int^x (q - r)\,\mathrm{d}x'\right),$$

they satisfy the system of coupled mKdV equations (4.62).

4.2.24. System (4.29). Through introduction of a new scalar variable w by

$$w \equiv u_x - u^2,$$

system (4.29) is transformed to a system that looks very similar to (4.64),

$$\begin{cases} w_t = w_{xxx} + 6ww_x + w_x\langle U, U\rangle + 2w\langle U, U\rangle_x + \frac{1}{2}\langle U, U\rangle_{xxx}, \\ U_t = U_{xxx} + 6(wU)_x + \langle U, U\rangle U_x + 2\langle U, U\rangle_x U. \end{cases} \tag{4.69}$$

System (4.69) is a multi-component generalization of a flow of the Jaulent–Miodek hierarchy [82]. Let us demonstrate that (4.69) admits a Lax representation. We consider a pair of linear equations for a column-vector function ψ,

$$\begin{cases} \psi_{xx} + (Q + \zeta R)\psi = \zeta^2\psi, \\ \psi_t = \left(4\zeta^2 I + 2\zeta R + 2Q + \frac{3}{2}R^2\right)\psi_x - \left[\zeta R_x + Q_x + \frac{3}{4}(R^2)_x\right]\psi. \end{cases} \tag{4.70}$$

Here, ζ is the spectral parameter, and Q and R are square matrices with the same dimension. The compatibility condition $\psi_{xxt} = \psi_{txx}$ for (4.70) implies a system of two matrix equations

$$\begin{cases} Q_t = Q_{xxx} + 3(Q^2)_x + \frac{3}{4}(R^2)_{xxx} + \frac{3}{2}R^2 Q_x + \frac{3}{4}[Q(R^2)_x + 3(R^2)_x Q], \\ R_t = R_{xxx} + 3(QR + RQ)_x + \frac{3}{4}[3(R^2)_x R + R(R^2)_x + 2R^2 R_x], \end{cases} \tag{4.71}$$

together with one constraint

$$[Q, R^2] = O.$$

If we consider the reduction

$$Q = w\mathbf{1}, \qquad R = \frac{\sqrt{6}}{3}\mathrm{i}\sum_{j=1}^{N} U_j \mathbf{e}_j, \qquad \{\mathbf{e}_i, \mathbf{e}_j\}_+ = -2\delta_{ij}\mathbf{1},$$

the constraint is automatically satisfied and system (4.71) collapses to system (4.69). This Lax representation for (4.69) can be rewritten as that for (4.29) [83].

Relation to system (4.17). If we introduce a new scalar variable v by

$$v \equiv u_x - u^2 + \frac{1}{6}\langle U, U\rangle, \tag{4.72}$$

system (4.29) is changed into the following system:

$$\begin{cases} v_t = \left(v_{xx} + 3v^2 + 2v\langle U, U\rangle + \langle U, U_{xx}\rangle + \frac{1}{2}\langle U_x, U_x\rangle\right)_x, \\ U_t = U_{xxx} + 6v_x U + 6v U_x + 2\langle U, U_x\rangle U. \end{cases} \tag{4.73}$$

Then, it is straightforward to obtain (4.17) (for $\hat{u}$ and U) from (4.73) through potentiation $v = \hat{u}_x$.

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = \tfrac{1}{2}(q + r), \qquad U = \tfrac{\sqrt{6}}{2}(q - r).$$

Then we can rewrite (4.29) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = [q_{xx} + 3(q - r)q_x + q^3 - 6q^2 r + 3qr^2]_x, \\ r_t = [r_{xx} - 3(q - r)r_x + 3q^2 r - 6qr^2 + r^3]_x. \end{cases}$$

This system coincides with (51) in [25] or (3.16) in [26]. It is known as a flow of the modified Jaulent–Miodek hierarchy [83] (see also (7.37) in [84]). While elaborating on this paper, the authors encountered one paper [85] on the three-component generalization of this flow ((4.29) with $N = 2$).

4.2.25. System (4.30). For system (4.30), if we define new variables w and W by

$$\begin{cases} w \equiv u_x + u^2 + \frac{1}{6}\langle U, U\rangle, \\ W \equiv U_x + 2uU, \end{cases} \tag{4.74}$$

they satisfy the following system:

$$\begin{cases} w_t = w_{xxx} - 6ww_x + 2\langle W, W_x\rangle, \\ W_t = -2W_{xxx} + 6wW_x. \end{cases} \tag{4.75}$$

This system coincides with the multi-component Hirota–Satsuma system (3.6), up to a scaling of variables. The Miura map (4.74) is a multi-component generalization of that for the case of scalar U in [43, 44] and that for the case of two-component vector U in [61].

Relation to system (4.18). If we introduce a new scalar variable v by

$$v \equiv u_x - u^2 - \tfrac{1}{6}\langle U, U\rangle, \tag{4.76}$$

system (4.30) is changed into the following system (cf (4.3) in [86]):

$$\begin{cases} v_t = \left(v_{xx} + 3v^2 + 4v\langle U, U\rangle + 2\langle U, U_{xx}\rangle + \langle U_x, U_x\rangle + \frac{2}{3}\langle U, U\rangle^2\right)_x, \\ U_t = -2U_{xxx} - 6v_x U - 6vU_x - 4\langle U, U_x\rangle U. \end{cases} \tag{4.77}$$

Then, it is straightforward to obtain (4.18) (for $\hat{u}$ and U) from (4.77) through potentiation $v = \hat{u}_x$. Combining (4.76) and (4.74), we obtain the relation $v + w = 2u_x$, and consequently,

$$\hat{u} + \int^x w \, \mathrm{d}x' = 2u.$$

Using this relation, we can also rewrite (4.74) as a transformation between system (4.18) and the multi-component Hirota–Satsuma system (4.75).

Symmetrization. In the case where U is scalar, we consider the linear change of variables

$$u = -\tfrac{1}{2}(q + r), \qquad U = \tfrac{\sqrt{6}}{2}\mathrm{i}(q - r).$$

Then we can rewrite (4.30) as a system of two symmetrically coupled mKdV equations

$$\begin{cases} q_t = \left[-\frac{1}{2}q_{xx} + \frac{3}{2}r_{xx} + 3(q - r)q_x - 2r^3\right]_x, \\ r_t = \left[\frac{3}{2}q_{xx} - \frac{1}{2}r_{xx} - 3(q - r)r_x - 2q^3\right]_x. \end{cases}$$

This system is identical to (63) in [25] or (3.22) in [26]. It was found in connection with the Kac–Moody Lie algebras and written in a Hamiltonian form about 20 years ago (cf the $C_2^{(1)}$ case in [43] or the $B_2^{(1)}$ case in [44]).

5. The case $\lambda_1 = \lambda_2 = \frac{1}{2}$: coupled Ibragimov–Shabat equations

In this section, we classify second-order and third-order systems in the $\lambda_1 = \lambda_2 = \frac{1}{2}$ (Ibragimov–Shabat weighting [36]) case. In the first part (section 5.1), we present a complete list of such systems with a specific order symmetry. In the second part (section 5.2), we prove that the listed systems are linearizable.

5.1. List of systems with a higher symmetry

The general ansatz for a $\lambda_1 = \lambda_2 = \frac{1}{2}$ homogeneous evolutionary system of second order for a scalar function u and a vector function U takes the form

$$\begin{cases} u_{t_2} = a_1 u_{xx} + a_2 u^2 u_x + a_3 u^5 + a_4 u_x \langle U, U\rangle + a_5 u \langle U, U_x\rangle \\ \qquad + a_6 u^3 \langle U, U\rangle + a_7 u \langle U, U\rangle^2, \\ U_{t_2} = a_8 U_{xx} + a_9 u u_x U + a_{10} u^2 U_x + a_{11} u^4 U + a_{12} \langle U, U\rangle U_x \\ \qquad + a_{13} \langle U, U_x\rangle U + a_{14} u^2 \langle U, U\rangle U + a_{15} \langle U, U\rangle^2 U. \end{cases} \tag{5.1}$$

The following constraints guarantee the order to be 2 and the system not to be triangular:

$$(a_1, a_8) \neq (0, 0), \qquad (a_4, a_5, a_6, a_7) \neq (0, 0, 0, 0), \qquad (a_9, a_{10}, a_{11}, a_{14}) \neq (0, 0, 0, 0).$$

Similarly, the general ansatz for a third-order system takes the form

$$\begin{cases} u_{t_3} = b_1 u_{xxx} + b_2 u^2 u_{xx} + b_3 u u_x^2 + b_4 u^4 u_x + b_5 u^7 + b_6 u_{xx} \langle U, U\rangle \\ \qquad + b_7 u_x \langle U, U_x\rangle + b_8 u \langle U_x, U_x\rangle + b_9 u \langle U, U_{xx}\rangle + b_{10} u^2 u_x \langle U, U\rangle \\ \qquad + b_{11} u^3 \langle U, U_x\rangle + b_{12} u^5 \langle U, U\rangle + b_{13} u_x \langle U, U\rangle^2 \\ \qquad + b_{14} u \langle U, U\rangle \langle U, U_x\rangle + b_{15} u^3 \langle U, U\rangle^2 + b_{16} u \langle U, U\rangle^3, \\ U_{t_3} = b_{17} U_{xxx} + b_{18} u u_{xx} U + b_{19} u_x^2 U + b_{20} u u_x U_x + b_{21} u^2 U_{xx} \\ \qquad + b_{22} u^3 u_x U + b_{23} u^4 U_x + b_{24} u^6 U + b_{25} \langle U, U\rangle U_{xx} + b_{26} \langle U, U_x\rangle U_x \\ \qquad + b_{27} \langle U_x, U_x\rangle U + b_{28} \langle U, U_{xx}\rangle U + b_{29} u u_x \langle U, U\rangle U \\ \qquad + b_{30} u^2 \langle U, U\rangle U_x + b_{31} u^2 \langle U, U_x\rangle U + b_{32} u^4 \langle U, U\rangle U \\ \qquad + b_{33} \langle U, U\rangle^2 U_x + b_{34} \langle U, U\rangle \langle U, U_x\rangle U + b_{35} u^2 \langle U, U\rangle^2 U \\ \qquad + b_{36} \langle U, U\rangle^3 U, \end{cases} \tag{5.2}$$

for which the following constraints guarantee the order to be 3 and the system not to be triangular: $(b_1, b_{17}) \neq (0, 0)$ and at least one of $b_6, \ldots, b_{16}$ and one of $b_{18}, \ldots, b_{24}$, $b_{29}, \ldots, b_{32}, b_{35}$ must not vanish. However, when we consider a third-order symmetry for a second-order system, we relax these constraints as follows (cf section 2): $(b_1, b_{17}) \neq (0, 0)$ and at least one of $b_1, \ldots, b_{16}$ and one of $b_{17}, \ldots, b_{36}$ must not vanish.

Proposition 5.1. *No second-order system of the form (5.1) with a third-order symmetry of the form (5.2) or a fourth-order symmetry exists.*

Theorem 5.2. *Any third-order system of the form (5.2) with a fifth-order symmetry has to coincide with either of the following two systems up to a scaling of t_3, x, u, U (we omit the* *subscript of t_3):*

$$\begin{cases} u_t = (a+1)(u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x + 3u_{xx}\langle U, U\rangle \\ \quad + 6u_x\langle U, U_x\rangle + 3u_x\langle U, U\rangle^2) + 2au\langle U, U_{xx}\rangle \\ \quad + (2a+3)u\langle U_x, U_x\rangle + (10a+6)u_xu^2\langle U, U\rangle + 2au^3\langle U, U_x\rangle \\ \quad + 6au\langle U, U\rangle\langle U, U_x\rangle + au^5\langle U, U\rangle + 2au^3\langle U, U\rangle^2 + au\langle U, U\rangle^3, \\ U_t = U_{xxx} + 3\langle U, U\rangle U_{xx} + 6\langle U, U_x\rangle U_x + 3\langle U_x, U_x\rangle U + 3\langle U, U\rangle^2 U_x \\ \quad - 2au_{xx}uU + (a+3)u_x^2U + 6uu_xU_x + 3u^2U_{xx} - 6au_xu^3U \\ \quad + 3u^4U_x - 2au_xu\langle U, U\rangle U - 4au^2\langle U, U_x\rangle U + 6u^2\langle U, U\rangle U_x \\ \quad - au^6U - 2au^4\langle U, U\rangle U - au^2\langle U, U\rangle^2 U, \qquad a \textit{ is arbitrary}, \end{cases} \tag{5.3}$$

$$\begin{cases} u_t = u_{xxx} + 3u^2u_{xx} + 9uu_x^2 + 3u^4u_x + 3u_{xx}\langle U, U\rangle + 6u_x\langle U, U_x\rangle \\ \quad + 2u\langle U, U_{xx}\rangle + 2u\langle U_x, U_x\rangle + 10u_xu^2\langle U, U\rangle + 2u^3\langle U, U_x\rangle \\ \quad + 3u_x\langle U, U\rangle^2 + 6u\langle U, U\rangle\langle U, U_x\rangle + u^5\langle U, U\rangle + 2u^3\langle U, U\rangle^2 + u\langle U, U\rangle^3, \\ U_t = -2u_{xx}uU + u_x^2U - 6u_xu^3U - 2u_xu\langle U, U\rangle U - 4u^2\langle U, U_x\rangle U \\ \quad - u^6U - 2u^4\langle U, U\rangle U - u^2\langle U, U\rangle^2 U. \end{cases} \tag{5.4}$$

Both system (5.3) and system (5.4) admit the reduction $U = \mathbf{0}$. From this viewpoint, they are considered as generalizations of the Ibragimov–Shabat equation [36]. In addition, system (5.3) admits the reduction $u = 0$ which changes it to a vector analogue of the Ibragimov–Shabat equation [15, 34],

$$U_t = U_{xxx} + 3\langle U, U\rangle U_{xx} + 6\langle U, U_x\rangle U_x + 3\langle U_x, U_x\rangle U + 3\langle U, U\rangle^2 U_x. \tag{5.5}$$

We can linearize (5.3) and (5.4) through the same change of dependent variables. In fact, both of them are third-order symmetries of a nontrivial first-order system,

$$\begin{cases} u_{t_1} = u_x + u\langle U, U\rangle, \\ U_{t_1} = -u^2U, \end{cases} \tag{5.6}$$

which is naturally linearizable in the same way.

5.2. *Integrability of systems (5.3) and (5.4)*

5.2.1. System (5.3). We note that system (5.3) possesses the following conservation law:

$$\begin{aligned} (u^2 + \langle U, U\rangle)_t = \big[&(a+1)\big(2uu_{xx} - u_x^2 + 6u^3u_x + u^6\big) + 2\langle U, U_{xx}\rangle - \langle U_x, U_x\rangle \\ &+ (4a+6)u^2\langle U, U_x\rangle + (2a+6)u_xu\langle U, U\rangle + (2a+3)u^4\langle U, U\rangle \\ &+ (a+3)u^2\langle U, U\rangle^2 + 6\langle U, U\rangle\langle U, U_x\rangle + \langle U, U\rangle^3\big]_x. \end{aligned} \tag{5.7}$$

Then, if we introduce new variables w and W by

$$\begin{cases} w \equiv u \exp\left(\int^x (u^2 + \langle U, U\rangle)\,\mathrm{d}x'\right), \\ W \equiv U \exp\left(\int^x (u^2 + \langle U, U\rangle)\,\mathrm{d}x'\right), \end{cases} \tag{5.8}$$

they satisfy a pair of linear equations

$$\begin{cases} w_t = (a+1)w_{xxx}, \\ W_t = W_{xxx}. \end{cases}$$

If we set $U = \mathbf{0}$ or $u = 0$, (5.8) is reduced to the linearizing transformation for the Ibragimov–Shabat equation [3, 87] and that for its vector analogue (5.5), respectively.

5.2.2. System (5.4). System (5.4) is obtained from (5.3) by rescaling t appropriately and taking the limit $a \to \infty$. As this fact implies in combination with (5.7), system (5.4) possesses the following conservation law:

$$(u^2 + \langle U, U\rangle)_t = \big(2uu_{xx} - u_x^2 + 6u^3u_x + u^6 + 2u_x u\langle U, U\rangle + 4u^2\langle U, U_x\rangle + 2u^4\langle U, U\rangle + u^2\langle U, U\rangle^2\big)_x.$$

Then, by the same change of variables as in section 5.2.1,

$$\begin{cases} w = u \exp\left(\int^x (u^2 + \langle U, U\rangle)\,\mathrm{d}x'\right), \\ W = U \exp\left(\int^x (u^2 + \langle U, U\rangle)\,\mathrm{d}x'\right), \end{cases}$$

system (5.4) is decoupled into one linear equation and one trivial equation

$$\begin{cases} w_t = w_{xxx}, \\ W_t = \mathbf{0}. \end{cases}$$

6. The case $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$: negative results

In this section, we search for second-order and third-order systems with a specific order symmetry in the case of $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$. However, the results turn out to be negative, as is shown below.

The general ansatz for a $\lambda_1 = \frac{1}{3}, \lambda_2 = \frac{2}{3}$ homogeneous evolutionary system of second order for a scalar function u and a vector function U takes the form

$$\begin{cases} u_{t_2} = a_1 u_{xx} + a_2 u^3 u_x + a_3 u^7 + a_4\langle U, U_x\rangle + a_5 u^3\langle U, U\rangle, \\ U_{t_2} = a_6 U_{xx} + a_7 u^2 u_x U + a_8 u^3 U_x + a_9 u^6 U + a_{10} u^2\langle U, U\rangle U. \end{cases} \tag{6.1}$$

The following constraints guarantee the order to be 2 and the system not to be triangular:

$$(a_1, a_6) \neq (0, 0), \qquad (a_4, a_5) \neq (0, 0), \qquad (a_7, a_8, a_9, a_{10}) \neq (0, 0, 0, 0).$$

Similarly, the general ansatz for a third-order system takes the form

$$\begin{cases} u_{t_3} = b_1 u_{xxx} + b_2 u^3 u_{xx} + b_3 u^2 u_x^2 + b_4 u^6 u_x + b_5 u^{10} + b_6\langle U, U_{xx}\rangle \\ \qquad + b_7\langle U_x, U_x\rangle + b_8 u^2 u_x\langle U, U\rangle + b_9 u^3\langle U, U_x\rangle + b_{10} u^6\langle U, U\rangle \\ \qquad + b_{11} u^2\langle U, U\rangle^2, \\ U_{t_3} = b_{12} U_{xxx} + b_{13} u^2 u_{xx} U + b_{14} u u_x^2 U + b_{15} u^2 u_x U_x + b_{16} u^3 U_{xx} \\ \qquad + b_{17} u^5 u_x U + b_{18} u^6 U_x + b_{19} u^9 U + b_{20} u u_x\langle U, U\rangle U \\ \qquad + b_{21} u^2\langle U, U\rangle U_x + b_{22} u^2\langle U, U_x\rangle U + b_{23} u^5\langle U, U\rangle U \\ \qquad + b_{24} u\langle U, U\rangle^2 U, \end{cases} \tag{6.2}$$

for which the following constraints guarantee the order to be 3 and the system not to be triangular: $(b_1, b_{12}) \neq (0, 0)$ and at least one of $b_6, \ldots, b_{11}$ and one of $b_{13}, \ldots, b_{24}$ must not vanish. However, when we consider a third-order symmetry for a second-order system, we relax these constraints as follows (cf section 2): $(b_1, b_{12}) \neq (0, 0)$ and at least one of $b_1, \ldots, b_{11}$ and one of $b_{12}, \ldots, b_{24}$ must not vanish.

Proposition 6.1. *No second-order system of the form (6.1) with a third-order symmetry of the form (6.2) or a fourth-order symmetry exists.*

Proposition 6.2. *No third-order system of the form (6.2) with a fifth-order symmetry exists.*

7. The case $\lambda_1 = \frac{2}{3}$, $\lambda_2 = \frac{1}{3}$

In this section, we classify second-order and third-order systems in the case of $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3}$. In the first part (section 7.1), we present complete lists of such systems with a specific order symmetry. In the second part (section 7.2), we prove that the listed systems are linearizable.

7.1. Lists of systems with a higher symmetry

The general ansatz for a $\lambda_1 = \frac{2}{3}, \lambda_2 = \frac{1}{3}$ homogeneous evolutionary system of second order for a scalar function u and a vector function U takes the form

$$\begin{cases} u_{t_2} = a_1 u_{xx} + a_2 u^4 + a_3 \langle U, U_{xx}\rangle + a_4\langle U_x, U_x\rangle + a_5 u^3 \langle U, U\rangle \\ \qquad + a_6 u^2 \langle U, U\rangle^2 + a_7 u \langle U, U\rangle^3 + a_8 \langle U, U\rangle^4, \\ U_{t_2} = a_9 U_{xx} + a_{10} u^3 U + a_{11} u^2 \langle U, U\rangle U + a_{12} u \langle U, U\rangle^2 U + a_{13}\langle U, U\rangle^3 U. \end{cases} \tag{7.1}$$

The following constraints guarantee the order to be 2 and the system not to be triangular:

$$(a_1, a_3, a_9) \neq (0,0,0), \qquad (a_3, a_4, a_5, a_6, a_7, a_8) \neq (0,0,0,0,0,0),$$
$$(a_{10}, a_{11}, a_{12}) \neq (0,0,0).$$

Similarly, the general ansatz for a third-order system takes the form

$$\begin{cases} u_{t_3} = b_1 u_{xxx} + b_2 u^3 u_x + b_3 \langle U, U_{xxx}\rangle + b_4 \langle U_x, U_{xx}\rangle + b_5 u^2 u_x \langle U, U\rangle \\ \qquad + b_6 u^3 \langle U, U_x\rangle + b_7 u u_x \langle U, U\rangle^2 + b_8 u^2 \langle U, U\rangle\langle U, U_x\rangle \\ \qquad + b_9 u_x \langle U, U\rangle^3 + b_{10} u \langle U, U\rangle^2 \langle U, U_x\rangle + b_{11}\langle U, U\rangle^3 \langle U, U_x\rangle, \\ U_{t_3} = b_{12} U_{xxx} + b_{13} u^2 u_x U + b_{14} u^3 U_x + b_{15} u u_x \langle U, U\rangle U \\ \qquad + b_{16} u^2 \langle U, U\rangle U_x + b_{17} u^2 \langle U, U_x\rangle U + b_{18} u_x \langle U, U\rangle^2 U \\ \qquad + b_{19} u \langle U, U\rangle^2 U_x + b_{20} u \langle U, U\rangle \langle U, U_x\rangle U + b_{21}\langle U, U\rangle^3 U_x \\ \qquad + b_{22}\langle U, U\rangle^2 \langle U, U_x\rangle U, \end{cases} \tag{7.2}$$

for which the following constraints guarantee the order to be 3 and the system not to be triangular: $(b_1, b_3, b_{12}) \neq (0,0,0)$ and at least one of $b_3, \ldots, b_{11}$ and one of $b_{13}, \ldots, b_{20}$ must not vanish. However, when we consider a third-order symmetry for a second-order system, we relax these constraints as follows (cf section 2): $(b_1, b_3, b_{12}) \neq (0,0,0)$ and at least one of $b_1, \ldots, b_{11}$ and one of $b_{12}, \ldots, b_{22}$ must not vanish.

Theorem 7.1. *Any second-order system of the form (7.1) with a third-order symmetry of the form (7.2) has to coincide with the following system up to a scaling of t_2, x, u, U (we omit the subscript of t_2):*

$$\begin{cases} u_t = u_{xx} + 2\langle U, U_{xx}\rangle + 2\langle U_x, U_x\rangle + 2u\langle U, U\rangle^3 + 2\langle U, U\rangle^4, \\ U_t = -u\langle U, U\rangle^2 U - \langle U, U\rangle^3 U. \end{cases} \tag{7.3}$$

Theorem 7.2. *Any second-order system of the form (7.1) with a fourth-order symmetry has to coincide with either (7.3) or the following system up to a scaling of t_2, x, u, U:*

$$\begin{cases} u_t = -2\langle U, U_{xx}\rangle - 2u^3\langle U, U\rangle - 6u^2\langle U, U\rangle^2 - 6u\langle U, U\rangle^3 - 2\langle U, U\rangle^4, \\ U_t = U_{xx} + u^3 U + 3u^2 \langle U, U\rangle U + 3u\langle U, U\rangle^2 U + \langle U, U\rangle^3 U. \end{cases} \tag{7.4}$$

Theorem 7.3. *Any third-order system of the form (7.2) with a fifth-order symmetry has to coincide with either of the following two systems up to a scaling of t_3, x, u, U (we omit the subscript of t_3):*

$$\begin{cases} u_t = u_{xxx} + 2\langle U, U_{xxx}\rangle + 6\langle U_x, U_{xx}\rangle + 2u_x\langle U, U\rangle^3 + 4\langle U, U\rangle^3\langle U, U_x\rangle, \\ U_t = -u_x\langle U, U\rangle^2 U - 2\langle U, U\rangle^2\langle U, U_x\rangle U, \end{cases} \tag{7.5}$$

$$\begin{cases} u_t = u_{xxx} + 2\langle U, U_{xxx}\rangle + 6\langle U_x, U_{xx}\rangle + 2u_x\langle U, U\rangle^3 + 4\langle U, U\rangle^3\langle U, U_x\rangle, \\ U_t = -u_x\langle U, U\rangle^2 U - 4u\langle U, U\rangle\langle U, U_x\rangle U + 4u\langle U, U\rangle^2 U_x \\ \qquad - 6\langle U, U\rangle^2\langle U, U_x\rangle U + 4\langle U, U\rangle^3 U_x. \end{cases} \tag{7.6}$$

We note that (7.5) is the third-order symmetry of the second-order system (7.3).

7.2. Integrability of systems (7.3)–(7.6)

7.2.1. Systems (7.3) and (7.5). We present a procedure for solving system (7.3) only, because its third-order symmetry (7.5) can be solved in the same way. For system (7.3), if we introduce a new variable w by

$$w \equiv u + \langle U, U\rangle, \tag{7.7}$$

it solves the linear equation

$$w_t = w_{xx}.$$

Once we know $w(x,t)$ by solving this equation, we obtain from the relation $(\langle U, U\rangle^{-2})_t = 4w$ that

$$\frac{1}{\langle U(x,t), U(x,t)\rangle^2} = 4\int_0^t w(x,t')\,\mathrm{d}t' + \frac{1}{\langle U(x,0), U(x,0)\rangle^2}.$$

Then we can determine $u(x,t)$ by using (7.7). Finally, noting the relation $\left(\langle U, U\rangle^{-\frac{1}{2}} U\right)_t = \mathbf{0}$, we obtain the following expression for $U(x,t)$:

$$U(x,t) = \frac{1}{\left[1 + 4\langle U(x,0), U(x,0)\rangle^2 \int_0^t w(x,t')\,\mathrm{d}t'\right]^{\frac{1}{4}}} U(x,0).$$

7.2.2. System (7.4). For system (7.4), we have the relation $(u + \langle U, U\rangle)_t = 0$. Thus, we can set

$$u + \langle U, U\rangle \equiv \phi(x),$$

where the function $\phi(x)$ does not depend on t. Then, the equation for U is rewritten in terms of $\phi(x)$ as

$$U_t = U_{xx} + \phi^3 U. \tag{7.8}$$

The solutions of (7.8) are given by

$$U(x,t) = \int \mathrm{d}\lambda\, \mathrm{e}^{\lambda t}\Psi(x;\lambda),$$

where $\Psi(x;\lambda)$ is a solution of the ordinary differential equation

$$\Psi_{xx} + \phi^3\Psi = \lambda\Psi.$$

The following commutation relation indicates that system (7.4) possesses a polynomial higher symmetry of every even order (cf [7, 10]):

$$\left[\partial_x^2 + \phi^3, (\partial_x^2 + \phi^3)^n\right] = 0, \qquad n = 1, 2, \ldots.$$

7.2.3. System (7.6). For system (7.6), if we introduce a new variable w by

$$w \equiv u + \langle U, U\rangle,$$

it solves the linear equation

$$w_t = w_{xxx}.$$

Once we know $w(x,t)$, we obtain from the relation $(\langle U, U\rangle^{-2})_t = 4w_x$ that

$$\frac{1}{\langle U(x,t), U(x,t)\rangle^2} = 4\int_0^t w_x(x,t')\,\mathrm{d}t' + \frac{1}{\langle U(x,0), U(x,0)\rangle^2}. \tag{7.9}$$

Then, the equation for $\langle U, U\rangle^{-\frac{1}{2}}U$ can be rewritten as

$$\left(\frac{1}{\sqrt{\langle U, U\rangle}}U\right)_t = 4(u + \langle U, U\rangle)\langle U, U\rangle^2\left(\frac{1}{\sqrt{\langle U, U\rangle}}U\right)_x$$
$$= \frac{4w(x,t)}{4\int_0^t w_x(x,t')\,\mathrm{d}t' + \frac{1}{\langle U(x,0),U(x,0)\rangle^2}}\left(\frac{1}{\sqrt{\langle U, U\rangle}}U\right)_x.$$

The general solution of this equation is given by

$$\frac{1}{\sqrt{\langle U, U\rangle}}U_j = f_j\left(4\int_0^t w(x,t')\,\mathrm{d}t' + \int^x \frac{1}{\langle U(x',0), U(x',0)\rangle^2}\,\mathrm{d}x'\right), \qquad j = 1, 2, \ldots, N, \tag{7.10}$$

where $f_1(z), \ldots, f_N(z)$ are arbitrary functions of z, except that they must satisfy one constraint, $\sum_{j=1}^N [f_j(z)]^2 = 1$. Combining (7.10) with (7.9), we arrive at the following formula:

$$U_j(x,t) = \frac{1}{(\xi_x)^{\frac{1}{4}}} f_j(\xi), \qquad j = 1, 2, \ldots, N,$$

where $\xi(x,t) \equiv 4\int_0^t w(x,t')\,\mathrm{d}t' + \int^x \langle U(x',0), U(x',0)\rangle^{-2}\,\mathrm{d}x'$.

8. Concluding remarks

In this paper, we have presented a classification of integrable evolutionary systems in 1+1 dimensions for one scalar unknown $u(x,t)$ and one vector unknown $U(x,t)$. We focused on polynomial systems that are homogeneous under a suitable weighting of ∂_x, ∂_t, $u(x,t)$, $U(x,t)$ and considered five distinct weightings for u, U relative to a fixed weight of ∂_x. Then, with the help of a computer algebra program, we obtained the complete lists, up to a scaling of variables, of second-order systems with a third-order or a fourth-order symmetry and third-order systems with a fifth-order symmetry. We demonstrated the integrability of nearly all listed systems by constructing a Lax representation or a linearizing transformation or, in some cases, by identifying an integrable closed subsystem contained in the system under investigation.

Table 4 gives a quick overview of the systems found. Note we use 'MT' as an abbreviation for 'Miura-type transformation', including Miura map plus potentiation. In table 4, we set the weight of ∂_x at unity, without any loss of generality. For full details regarding Lax representations, transformations, references, etc, the reader is referred to the corresponding part of the paper identified through the equation number. Here, we would like to make a few remarks on our classification results:

- The most interesting classification results are obtained for the case $\lambda_1 = \lambda_2 = 1$, namely the Burgers/pKdV/mKdV weighting. The lists in this case consist of a large number of systems, which are shown to have a very wide variety of underlying structures. We compared these lists thoroughly with the lists of two-component systems by Foursov–Olver [18, 25, 26], refined and generalized their work, as described in the introduction.

Table 4. An overview of the considered classes with unit weighting of ∂_x.

Weights (λ_1, λ_2) of u, U	Weights of $\partial_t, \partial_\tau$ in sys., sym.	System	Comments
(2, 2)	2, 3	None	
	2, 4	None	
	3, 5	(3.3) $\begin{cases} u_t = \langle U, U_x\rangle, \\ U_t = U_{xxx} + u_x U + 2uU_x \end{cases}$	multi-component generalization of a Drinfel'd–Sokolov system [43, 44], see [45–47]
		(3.4) $\begin{cases} u_t = u_{xxx} + 6uu_x - 6\langle U, U_x\rangle, \\ U_t = U_{xxx} + 6u_x U + 6uU_x \end{cases}$	known as a Jordan KdV system [27, 28, 34, 48]
		(3.5) $\begin{cases} u_t = u_{xxx} + 3uu_x + 3\langle U, U_x\rangle, \\ U_t = u_x U + uU_x \end{cases}$	multi-component generalization of Zakharov–Ito system [52, 53], see [54]
		(3.6) $\begin{cases} u_t = u_{xxx} + 6uu_x - 12\langle U, U_x\rangle, \\ U_t = -2U_{xxx} - 6uU_x \end{cases}$	multi-component generalization of Hirota–Satsuma system [57], see [58]
(1, 1)	$\begin{cases} 2, 3 \\ \text{or} \\ 2, 4 \end{cases}$	(4.3) $\begin{cases} u_t = \frac{1}{3}(1+2a)(u_{xx} + 2uu_x) + \frac{4}{3}\langle U, U_x\rangle, \\ U_t = U_{xx} + \frac{1}{3}(1-a)u_x U + uU_x \\ \quad + \frac{1}{12}(1-4a)u^2 U - \frac{1}{3}\langle U, U\rangle U, \\ \quad a \text{ is arbitrary} \end{cases}$	linearized by an extended Hopf–Cole transformation
		(4.4) $\begin{cases} u_t = u_{xx} + 2uu_x + 2\langle U, U_x\rangle, \\ U_t = -\frac{1}{2}u_x U - \frac{1}{2}u^2 U - \frac{1}{2}\langle U, U\rangle U \end{cases}$	is scaling limit of (4.3), linearized by the same transformation
		(4.5) $\begin{cases} u_t = u_{xx} + 2uu_x + \langle U, U_x\rangle, \\ U_t = \frac{1}{2}u_x U + uU_x \end{cases}$	contains two-component Burgers system (4.35) as closed subsystem, *integrability unproven*
	3, 5	(4.6) $\begin{cases} u_t = a(u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2 u_x) \\ \quad + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + 2\langle U, U_{xx}\rangle \\ \quad + 2\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + \frac{1}{2}(1-a)u_{xx}U + \frac{3}{2}u_x U_x \\ \quad + \frac{3}{2}uU_{xx} + \frac{3}{4}(1-2a)uu_x U + \frac{3}{4}u^2 U_x \\ \quad - \langle U, U_x\rangle U + \frac{1}{8}(1-4a)u^3 U \\ \quad - \frac{1}{2}u\langle U, U\rangle U, \quad a \text{ is arbitrary} \end{cases}$	is symmetry of (4.3)
		(4.7) $\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2 u_x \\ \quad + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + 2\langle U, U_{xx}\rangle \\ \quad + 2\langle U_x, U_x\rangle, \\ U_t = -\frac{1}{2}u_{xx}U - \frac{3}{2}uu_x U - \langle U, U_x\rangle U \\ \quad - \frac{1}{2}u^3 U - \frac{1}{2}u\langle U, U\rangle U \end{cases}$	is symmetry of (4.4), scaling limit of (4.6)
		(4.8) $\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2 u_x \\ \quad + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle \\ \quad + \langle U_x, U_x\rangle, \\ U_t = \frac{1}{2}u_{xx}U + u_x U_x + uu_x U + u^2 U_x \\ \quad + \frac{1}{2}\langle U, U\rangle U_x + \frac{1}{2}\langle U, U_x\rangle U \end{cases}$	is symmetry of (4.5), see there
		(4.9) $\begin{cases} u_t = u_{xxx} + 3uu_{xx} + 3u_x^2 + 3u^2 u_x \\ \quad + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle \\ \quad + \langle U_x, U_x\rangle, \\ U_t = \frac{1}{2}u_{xx}U + u_x U_x + uu_x U + u^2 U_x \\ \quad + \langle U, U\rangle U_x \end{cases}$	contains third-order symmetry of two-component Burgers system (4.35), *integrability unproven*
		(4.10) $\begin{cases} u_t = 3u_x\langle U, U\rangle + 3\langle U, U_{xx}\rangle - 3\langle U, U\rangle^2, \\ U_t = U_{xxx} + u_{xx}U + u_x U_x - 3\langle U, U_x\rangle U \end{cases}$	obtained from (4.12) by MT
		(4.11) $\begin{cases} u_t = 2u_x\langle U, U\rangle + 2\langle U, U_{xx}\rangle \\ \quad - \langle U_x, U_x\rangle - 2\langle U, U\rangle^2, \\ U_t = U_{xxx} + u_{xx}U + 2u_x U_x \\ \quad - 2\langle U, U\rangle U_x - 2\langle U, U_x\rangle U \end{cases}$	linearizable by change of variable, related to Kaup–Kupershmidt equation in a certain manner

Table 4. (Continued.)

Weights (λ_1, λ_2) of u, U	Weights of $\partial_t, \partial_\tau$ in sys., sym.	System	Comments
(1, 1)	3, 5	(4.12) $\begin{cases} u_t = u_x\langle U, U\rangle + 2u\langle U, U_x\rangle + \langle U, U_{xx}\rangle \\ \quad + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + u_{xx}U + u_xU_x - 2uu_xU \\ \quad - u^2U_x + \langle U, U\rangle U_x - \langle U, U_x\rangle U \end{cases}$	connected with (3.3) and (4.10) by MT
		(4.13) $\begin{cases} u_t = u_{xxx} + \frac{3}{2}u_x^2 + \frac{3}{2}\langle U_x, U_x\rangle, \\ U_t = u_xU_x \end{cases}$	is potential form of (3.5)
		(4.14) $\begin{cases} u_t = u_{xxx} + 3u_x^2 + 2au_x\langle U, U\rangle \\ \quad + a\langle U, U_{xx}\rangle + a\langle U_x, U_x\rangle + b\langle U, U\rangle^2, \\ U_t = u_{xx}U + 2u_xU_x + a\langle U, U\rangle U_x \\ \quad + a\langle U, U_x\rangle U, \quad (a, b) \neq (0, 0) \end{cases}$	for $b \neq a^2/4$, transformed to (3.5); for $b = a^2/4$, to KdV equation + linear vector equation coupled to it
		(4.15) $\begin{cases} u_t = u_{xxx} + 3u_x^2 - 3\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_xU_x \end{cases}$	is potential form of (3.4), MT connects to (4.27)
		(4.16) $\begin{cases} u_t = u_{xxx} + 3u_x^2 + u_x\langle U, U\rangle + \langle U, U_{xx}\rangle, \\ U_t = U_{xxx} + 3u_{xx}U + 3u_xU_x + \langle U, U_x\rangle U \end{cases}$	obtained from (4.28) by MT
		(4.17) $\begin{cases} u_t = u_{xxx} + 3u_x^2 + 2u_x\langle U, U\rangle + \langle U, U_{xx}\rangle \\ \quad + \frac{1}{2}\langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_{xx}U + 6u_xU_x + 2\langle U, U_x\rangle U \end{cases}$	obtained from (4.29) by MT
		(4.18) $\begin{cases} u_t = u_{xxx} + 3u_x^2 + 4u_x\langle U, U\rangle + 2\langle U, U_{xx}\rangle \\ \quad + \langle U_x, U_x\rangle + \frac{2}{3}\langle U, U\rangle^2, \\ U_t = -2U_{xxx} - 6u_{xx}U - 6u_xU_x \\ \quad - 4\langle U, U_x\rangle U \end{cases}$	obtained from (4.30) by MT
		(4.19) $\begin{cases} u_t = u_{xxx} + u_x^2 - 12\langle U, U_{xx}\rangle \\ \quad + 12\langle U_x, U_x\rangle - 4\langle U, U\rangle^2, \\ U_t = 4U_{xxx} + u_{xx}U + 2u_xU_x \\ \quad + 4\langle U, U\rangle U_x + 4\langle U, U_x\rangle U \end{cases}$	reduced to triangular system: KdV equation + linear vector equation coupled to it
		(4.20) $\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle \\ \quad + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x + \frac{3}{2}\langle U, U\rangle U_x \end{cases}$	converted to triangular system: KdV equation + nonlinear equation with interesting reduction (4.55) + linear vector equation
		(4.21) $\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle \\ \quad + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x + \frac{1}{2}\langle U, U\rangle U_x \\ \quad + \langle U, U_x\rangle U \end{cases}$	exactly like for (4.20) only different linear vector equation
		(4.22) $\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{1}{2}u_x\langle U, U\rangle \\ \quad + u\langle U, U_x\rangle + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = u_{xx}U + u_xU_x - uu_xU - \frac{1}{2}u^2U_x \\ \quad + \frac{1}{2}\langle U, U\rangle U_x + \langle U, U_x\rangle U \end{cases}$	obtained in [69], converted to triangular system: KdV equation + linear equations coupled to it, admits deformation connected with (3.5) by MT
		(4.23) $\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_x\langle U, U\rangle + u\langle U, U_x\rangle \\ \quad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle + \frac{1}{2}\langle U, U\rangle^2, \\ U_t = -u_xU_x - \frac{1}{2}u^2U_x - \frac{1}{2}\langle U, U\rangle U_x \\ \quad + \frac{1}{2}u\langle U, U\rangle U \end{cases}$	contains interesting triangular system (4.58) as closed subsystem: KdV equation + nonlinear equation coupled to it
		(4.24) $\begin{cases} u_t = u_{xxx} - \frac{3}{2}u^2u_x + u_x\langle U, U\rangle + u\langle U, U_x\rangle \\ \quad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle - \frac{1}{4}u^2\langle U, U\rangle \\ \quad + \frac{1}{4}\langle U, U\rangle^2, \\ U_t = \frac{1}{2}u_{xx}U + \frac{1}{2}\langle U, U_x\rangle U - \frac{1}{4}u^3U \\ \quad + \frac{1}{4}u\langle U, U\rangle U \end{cases}$	is symmetry of first-order system (4.61), converted to triangular system: KdV equation + Riccati equation coupled to it + linear vector equation coupled to them
		(4.25) $\begin{cases} u_t = u_{xxx} + u^2u_x + u_x\langle U, U\rangle, \\ U_t = U_{xxx} + u^2U_x + \langle U, U\rangle U_x \end{cases}$	equivalent to a single vector mKdV equation for vector (u, U)

Table 4. (Continued.)

Weights (λ_1, λ_2) of u, U	Weights of $\partial_t, \partial_\tau$ in sys., sym.	System	Comments
(1, 1)	3, 5	(4.26) $\begin{cases} u_t = u_{xxx} + 2u^2 u_x + u_x\langle U, U\rangle + u\langle U, U_x\rangle, \\ U_t = U_{xxx} + uu_x U + u^2 U_x + \langle U, U\rangle U_x \\ \quad + \langle U, U_x\rangle U \end{cases}$	equivalent to a single vector mKdV equation for vector (u, U)
		(4.27) $\begin{cases} u_t = u_{xxx} - 6u^2 u_x + 6u_x\langle U, U\rangle + 12u\langle U, U_x\rangle, \\ U_t = U_{xxx} - 12uu_x U - 6u^2 U_x + 6\langle U, U\rangle U_x \end{cases}$	known as a Jordan mKdV system [27], connected with (3.4) and (4.15) by MT
		(4.28) $\begin{cases} u_t = u_{xxx} - 6u^2 u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \quad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 3u_{xx} U + 3u_x U_x - 6uu_x U \\ \quad - 3u^2 U_x + \langle U, U\rangle U_x + 3\langle U, U_x\rangle U \end{cases}$	admits Lax representation, connected with (4.16) by MT
		(4.29) $\begin{cases} u_t = u_{xxx} - 6u^2 u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \quad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = U_{xxx} + 6u_{xx} U + 6u_x U_x - 12uu_x U \\ \quad - 6u^2 U_x + \langle U, U\rangle U_x + 4\langle U, U_x\rangle U \end{cases}$	multi-component generalization of a modified Jaulent–Miodek flow [83], admits Lax representation, connected with (4.17) by MT
		(4.30) $\begin{cases} u_t = u_{xxx} - 6u^2 u_x + u_x\langle U, U\rangle + 2u\langle U, U_x\rangle \\ \quad + \langle U, U_{xx}\rangle + \langle U_x, U_x\rangle, \\ U_t = -2U_{xxx} - 6u_{xx} U - 6u_x U_x + 12uu_x U \\ \quad + 6u^2 U_x + \langle U, U\rangle U_x - 2\langle U, U_x\rangle U \end{cases}$	connected with (3.6) and (4.18) by MT
$\left(\frac{1}{2}, \frac{1}{2}\right)$	2, 3	None	
	2, 4	None	
	3, 5	(5.3) $\begin{cases} u_t = (a+1)(u_{xxx} + 3u^2 u_{xx} + 9uu_x^2 + 3u^4 u_x \\ \quad + 3u_{xx}\langle U, U\rangle + 6u_x\langle U, U_x\rangle + 3u_x\langle U, U\rangle^2) \\ \quad + 2au\langle U, U_{xx}\rangle + (2a+3)u\langle U_x, U_x\rangle \\ \quad + (10a+6)u_x u^2\langle U, U\rangle + 2au^3\langle U, U_x\rangle \\ \quad + 6au\langle U, U\rangle\langle U, U_x\rangle + au^5\langle U, U\rangle \\ \quad + 2au^3\langle U, U\rangle^2 + au\langle U, U\rangle^3, \\ U_t = U_{xxx} + 3\langle U, U\rangle U_{xx} + 6\langle U, U_x\rangle U_x \\ \quad + 3\langle U_x, U_x\rangle U + 3\langle U, U\rangle^2 U_x - 2au_{xx} uU \\ \quad + (a+3)u_x^2 U + 6uu_x U_x + 3u^2 U_{xx} - 6au_x u^3 U \\ \quad + 3u^4 U_x - 2au_x u\langle U, U\rangle U - 4au^2\langle U, U_x\rangle U \\ \quad + 6u^2\langle U, U\rangle U_x - au^6 U - 2au^4\langle U, U\rangle U \\ \quad - au^2\langle U, U\rangle^2 U, \quad a \text{ is arbitrary} \end{cases}$	is symmetry of first-order system (5.6), extension of vector Ibragimov–Shabat equation, linearizable by change of variables
		(5.4) $\begin{cases} u_t = u_{xxx} + 3u^2 u_{xx} + 9uu_x^2 + 3u^4 u_x \\ \quad + 3u_{xx}\langle U, U\rangle + 6u_x\langle U, U_x\rangle + 2u\langle U, U_{xx}\rangle \\ \quad + 2u\langle U_x, U_x\rangle + 10u_x u^2\langle U, U\rangle + 2u^3\langle U, U_x\rangle \\ \quad + 3u_x\langle U, U\rangle^2 + 6u\langle U, U\rangle\langle U, U_x\rangle + u^5\langle U, U\rangle \\ \quad + 2u^3\langle U, U\rangle^2 + u\langle U, U\rangle^3, \\ U_t = -2u_{xx} uU + u_x^2 U - 6u_x u^3 U \\ \quad - 2u_x u\langle U, U\rangle U - 4u^2\langle U, U_x\rangle U \\ \quad - u^6 U - 2u^4\langle U, U\rangle U - u^2\langle U, U\rangle^2 U \end{cases}$	is symmetry of first-order system (5.6), scaling limit of (5.3), linearized by the same change of variables
$\left(\frac{1}{3}, \frac{2}{3}\right)$	2, 3	None	
	2, 4	None	
	3, 5	None	
$\left(\frac{2}{3}, \frac{1}{3}\right)$	2, 3	(7.3) $\begin{cases} u_t = u_{xx} + 2\langle U, U_{xx}\rangle + 2\langle U_x, U_x\rangle + 2u\langle U, U\rangle^3 \\ \quad + 2\langle U, U\rangle^4, \\ U_t = -u\langle U, U\rangle^2 U - \langle U, U\rangle^3 U \end{cases}$	ultralocal change of variables gives linear equations
	2, 4	(7.3)	

Table 4. (Continued.)

Weights (λ_1, λ_2) of u, U	Weights of $\partial_t, \partial_\tau$ in sys., sym.	System	Comments
$\left(\frac{2}{3}, \frac{1}{3}\right)$	2, 4	(7.4) $\begin{cases} u_t = -2\langle U, U_{xx}\rangle - 2u^3\langle U, U\rangle - 6u^2\langle U, U\rangle^2 \\ \quad - 6u\langle U, U\rangle^3 - 2\langle U, U\rangle^4, \\ U_t = U_{xx} + u^3 U + 3u^2\langle U, U\rangle U + 3u\langle U, U\rangle^2 U \\ \quad + \langle U, U\rangle^3 U \end{cases}$	ultralocal change of variables gives linear equations
	3, 5	(7.5) $\begin{cases} u_t = u_{xxx} + 2\langle U, U_{xxx}\rangle + 6\langle U_x, U_{xx}\rangle \\ \quad + 2u_x\langle U, U\rangle^3 + 4\langle U, U\rangle^3\langle U, U_x\rangle, \\ U_t = -u_x\langle U, U\rangle^2 U - 2\langle U, U\rangle^2\langle U, U_x\rangle U \end{cases}$	is symmetry of (7.3)
		(7.6) $\begin{cases} u_t = u_{xxx} + 2\langle U, U_{xxx}\rangle + 6\langle U_x, U_{xx}\rangle \\ \quad + 2u_x\langle U, U\rangle^3 + 4\langle U, U\rangle^3\langle U, U_x\rangle, \\ U_t = -u_x\langle U, U\rangle^2 U - 4u\langle U, U\rangle\langle U, U_x\rangle U \\ \quad + 4u\langle U, U\rangle^2 U_x - 6\langle U, U\rangle^2\langle U, U_x\rangle U \\ \quad + 4\langle U, U\rangle^3 U_x \end{cases}$	ultralocal change of variables gives linear equations

- We found a number of pairs/triplets of scalar–vector systems connected through transformations of dependent variables. Besides standard Miura transformations that map both the scalar and vector variables to new ones (see e.g. (4.44)), we also found Miura-type transformations that act only on the scalar variable and do not change the vector variable (see e.g. (4.46) combined with potentiation $v = \hat{u}_x$). For some other systems, we showed that a new scalar variable defined in terms of the old scalar and vector variables satisfies a closed integrable equation, such as the KdV equation or a linear equation.
- The search for such transformations in our case of scalar–vector systems is simple in comparison to scalar–scalar systems. For instance, the ansatz that a new scalar variable depends on the original vector variable U only through scalar products $\langle \partial_x^m U, \partial_x^n U\rangle$ narrows down the candidates for such transformations considerably. This leads us to the counter-intuitive observation that scalar–vector systems are, in a sense, more tractable than scalar–scalar systems. This is probably one reason why, unlike our work, Foursov and Olver proved integrability[18] for only a small proportion of their two-component systems [18, 25, 26].

  Finally, we mention some problems not solved in this paper:

- How can the integrability of the three systems (4.5), (4.8) and (4.9) be established along the lines of this paper? The main obstacle is that we know neither a linearizing transformation nor a proper Lax representation for the two-component Burgers system (4.35). The dependence of the functional form of travelling-wave solutions on the boundary conditions and the velocity implies that (4.35) is a highly nontrivial system and not linearizable by a naive extension of the Hopf–Cole transformation.
- Some scalar–vector systems are converted to a triangular form, i.e. a closed subsystem plus remaining equations coupled to it. When the remaining equations contain a nonlinear PDE in its own variable (cf (4.54*b*) or (4.58*b*)), it seems to be especially difficult to solve them explicitly for a given solution of the subsystem. Is there any method, like an extension of the inverse scattering method, for dealing with such triangular systems analytically?

[18] They discussed the existence of a recursion operator or a bi-Hamiltonian structure.

- Can one construct an explicit formula for the general solution of (4.55)? This equation is obtained from system (4.20) or (4.21) by converting it to a triangular form and then considering the special case in which the solution of the subsystem, KdV equation in this case, is identically zero.

Although we concentrated our attention on the five distinct weightings for u, U in this paper, we also found integrable systems that are homogeneous under a different weighting of variables. Namely, we obtained systems of coupled KdV–mKdV type, e.g. (4.47), (4.64), (4.69) and (4.77),[19] together with the proof of their integrability. We are planning to complete a classification of integrable systems of this type, i.e. scalar–vector systems with weights $\lambda_1 = 2, \lambda_2 = 1$, in a subsequent paper. Preliminary results can be viewed on the Web page http://lie.math.brocku.ca/twolf/htdocs/sv/over.html.

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